

Reproducing Kernel Interpolation Without Finite Element Enrichment

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ABSTRACT

A general formulation for developing reproducing kernel interpolation is presented. This is based on the coupling of a primitive function and an enrichment function. The primitive function introduces discrete Kronecker delta properties, while the enrichment function constitutes reproducing conditions. A necessary condition for obtaining a reproducing kernel interpolation function is an orthogonality condition between the vector of enrichment functions and the vector of shifted monomial functions at the discrete points. A normalized kernel function with relative small support is employed as the primitive function. This approach does not employ a finite element shape function and therefore the interpolation function can be arbitrarily smooth. To maintain the convergence properties of the original reproducing kernel approximation, a mixed interpolation is introduced. A rigorous error analysis is provided for the proposed method. Optimal order error estimates are shown for the meshfree interpolation in any Sobolev norms. Optimal order convergence is maintained when the proposed method is employed to solve one-dimensional boundary value problems. Numerical experiments are done demonstrating the theoretical error estimates. The performance of the method is illustrated in several sample problems.

1. Introduction

Meshfree shape functions developed from moving least squares approximation [1-4], partition of unity [5-7], or reproducing kernel approximation [8-10] are in general not interpolation functions. Since the approximation spaces formed by these approximation functions are not kinematically admissible, a Lagrange multiplier method [2] or a penalty method [11] is required in the Galerkin framework to impose essential boundary conditions. Efforts have been devoted to construct kinematically admissible test and trial function spaces for Galerkin meshfree methods. A transformation method that transforms generalized coordinates to nodal coordinates for degrees of freedom associated with essential boundaries has been introduced [12], recovering nodal values by introducing singularity to the kernel functions has been discussed [13, 12], and employment of D'Alembert's principle in conjunction with coordinate transformation has also been proposed for essential boundary condition treatment [14].

Interpolation properties in finite elements can be recovered in meshfree approximation in numerous ways. Examples are coupling meshfree and finite element shape functions with a ramping in the transition zone [15], coupling and enrichment of finite element and meshfree shape functions via enforcement of reproducing conditions [16], and hierarchical enrichment of finite element solution using a meshfree approximation [17]. Rigorous convergence analysis and error estimation of this latter technique is given in [18]. All these methods allow a direct imposition of essential boundary conditions, but requires an underlined grid structure for construction of finite element shape functions. While attempts have been made to develop stabilized nodal integration methods [19-21] for integration of weak form, it is desirable to formulate a meshfree interpolation function that does not require an underlined grid.

In the method of enriching lower order finite element solution by meshfree approximation [15], a set of finite element shape functions was first constructed in the designated sub-domains. Coupled moving least-squares shape functions are then formed by imposing reproducing conditions. If finite element nodes are different from meshfree nodes, separate coefficients for finite element and meshfree shape functions are needed. In the case where the meshfree nodes coincide with the finite element nodes, the moving least-squares approximation has to be at least one order higher to that of the intrinsic completeness of the FEM shape functions, and consequently, larger support sizes for moving least-squares shape functions are necessary.

This work presents a general formulation for a meshfree approximation that recovers nodal values at designated points without the finite element enrichment. The proposed method follows the idea of coupling meshfree and finite element shape functions by Belytschko et al. [15] and Huerta et al. [16]. In the current approach the meshfree approximation constitutes a primitive function and an enrichment function. The requirement in the enrichment function for recovering interpolation properties in the reproducing kernel approximation is first identified. A simple normalized kernel function that does not cover any neighbor points is used as the primitive function. For the purpose of simplifying imposition of essential boundary conditions while maintaining the convergence properties of the regular reproducing kernel approximation, a mixed interpolation is introduced. Primitive functions are only added to the essential boundary points so that the coefficients of meshfree approximation associated with the essential boundaries are nodal values.

In this paper, the reproducing kernel approximation is reviewed in Section 2. The first part of Section 3 discusses the general requirements for obtaining interpolation properties in the enriched reproducing kernel approximation, and the second part of this section presents specific forms of primitive function and enrichment functions. A mixed reproducing kernel interpolation is also discussed in this section. Rigorous convergence analysis and error estimate of the proposed method are presented in Section 4, followed by numerical verifications in Section 5. Several example problems are given in Section 6 to evaluate the performance of the reproducing kernel interpolation functions. Conclusions and remarks are given in Section 7.

2. Reproducing Kernel Approximation

The approximation methods widely used in meshfree methods are the moving least-squares (MLS) approximation [1-4], partition of unity method (PUM) [5-7], and the reproducing kernel (RK) approximation [8-10]. Since the reproducing conditions play an important role in the forthcoming discussion, the RK approximation is introduced in this section as the basis for developing meshfree interpolation functions.

To make the discussion general, we describe and analyze the method for an arbitrary d -dimensional problem, $d \geq 1$. For this purpose, we introduce some notations used in the rest of this paper. Let $\Omega \subset R^d$ be a nonempty, open bounded set with a Lipschitz continuous boundary. In the one-dimensional case, $d = 1$ and Ω is an open bounded interval. A generic point in R^d is denoted $\mathbf{x} = [x_1, \dots, x_d]^T$, or $\mathbf{y} = [y_1, \dots, y_d]^T$, or $\mathbf{z} = [z_1, \dots, z_d]^T$. We use the Euclidean norm to measure the vector length $\|\mathbf{x}\| = (\sum_{i=1}^d |x_i|^2)^{1/2}$. For $\mathbf{x}_0 \in R^d$ and $r > 0$,

denote $B_r(\mathbf{x}_0) = \{\mathbf{x} \in R^d \mid \|\mathbf{x} - \mathbf{x}_0\| \leq r\}$ for the closed ball with radius r centered at \mathbf{x}_0 . In particular, when $\mathbf{x}_0 = \mathbf{0}$, we write B_r for $B_r(\mathbf{0})$. We use n for the highest degree of the polynomials that can be reproduced by the meshfree methods. Let $P_n = P_n(\Omega)$ be the space of the polynomials of degree $\leq n$ on Ω . The dimension of the space P_n is $N_n = (n+d)!/(n!d!)$.

It is convenient to use the multi-dimensional notation $\alpha = (\alpha_1, \dots, \alpha_d)$ with $d \geq 0$ being

integers. The quantity $|\alpha| = \sum_{i=1}^d \alpha_i$ is the length of α . This allows a simple notation for partial

derivatives: $D^\alpha u(\mathbf{x}) = \frac{\partial^{|\alpha|} u(\mathbf{x})}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$. Denote $\alpha! = \alpha_1! \dots \alpha_d!$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$.

Consider a problem domain Ω discretized by a set of points $\{\mathbf{x}_I, \mathbf{x}_I \in \Omega\}_{I=1}^{NP}$. A function $u(\mathbf{x})$ defined in Ω is approximated by

$$u^h(\mathbf{x}) = \sum_I \left[\sum_{|\alpha| \leq n} (\mathbf{x} - \mathbf{x}_I)^\alpha b_\alpha(\mathbf{x}) \right] \Phi_{a_I}(\mathbf{x} - \mathbf{x}_I) u_I \quad (2.1)$$

where $\{(\mathbf{x} - \mathbf{x}_I)^\alpha\}_{|\alpha| \leq n}$ is a set of monomial basis functions and $b_\alpha(\mathbf{x})$, $|\alpha| \leq n$, are the coefficients of the basis functions that vary with the location of approximation \mathbf{x} . The function $\Phi_{a_I}(\mathbf{x} - \mathbf{x}_I)$ is the kernel function centered at \mathbf{x}_I that has a compact support $\omega_I = \text{Supp}(\Phi_{a_I}(\mathbf{x} - \mathbf{x}_I))$, $\Phi_{a_I}(\mathbf{x} - \mathbf{x}_I) > 0$ for $\mathbf{x} \in \omega_I$, and $\cup_I \omega_I \supset \Omega$. The support size a_I is a normalization parameter for distance measure so that $\Phi_{a_I}(\mathbf{x} - \mathbf{x}_I)$ vanishes when $\|\mathbf{x} - \mathbf{x}_I\| / a_I > 1$. For convenience, define the following notations:

$$\begin{aligned} \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) &= \{(\mathbf{x} - \mathbf{x}_I)^\alpha\}_{|\alpha| \leq n} \\ &= \{1, x_I - x_{II}, \dots, x_d - x_{dI}, (x_I - x_{II})^2, \dots, (x_d - x_{dI})^n\} \end{aligned} \quad (2.2)$$

$$\begin{aligned}\mathbf{b}^T(\mathbf{x}) &= \{b_\alpha(\mathbf{x})\}_{|\alpha| \leq n} \\ &= \{b_{0\dots 0}(\mathbf{x}), b_{10\dots 0}(\mathbf{x}), \dots, b_{0\dots 0I}(\mathbf{x}), b_{20\dots 0}(\mathbf{x}), \dots, b_{00\dots n}(\mathbf{x})\}\end{aligned}\quad (2.3)$$

and rewrite Eq. (2.1) to

$$u^h(\mathbf{x}) = \sum_I \Psi_I(\mathbf{x}) u_I \quad (2.4)$$

$$\Psi_I(\mathbf{x}) = \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{b}(\mathbf{x}) \Phi_{a_I}(\mathbf{x} - \mathbf{x}_I) \quad (2.5)$$

The coefficient vector $\mathbf{b}(\mathbf{x})$ is determined from the following reproducing conditions:

$$\sum_I \Psi_I(\mathbf{x}) \mathbf{x}_I^\alpha = \mathbf{x}^\alpha, \quad |\alpha| \leq n \quad (2.6)$$

Equation (2.6) is equivalent to

$$\sum_I \Psi_I(\mathbf{x}) (\mathbf{x} - \mathbf{x}_I)^\alpha = \delta_{|\alpha|, 0}, \quad |\alpha| \leq n \quad (2.7)$$

or

$$\sum_I \Psi_I(\mathbf{x}) \mathbf{H}(\mathbf{x} - \mathbf{x}_I) = \mathbf{H}(\mathbf{0}) \quad (2.8)$$

The $\mathbf{b}(\mathbf{x})$ vector is obtained by substituting Eq. (2.5) into Eq. (2.8) to yield

$$\mathbf{b}(\mathbf{x}) = \mathbf{M}^{-1}(\mathbf{x})\mathbf{H}(\mathbf{0}) \quad (2.9)$$

$$\mathbf{M}(\mathbf{x}) = \sum_I \mathbf{H}(\mathbf{x} - \mathbf{x}_I) \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \Phi_{a_I}(\mathbf{x} - \mathbf{x}_I) \quad (2.10)$$

Using Eqns. (2.5) and (2.9), function $\Psi_I(\mathbf{x})$ is obtained by

$$\Psi_I(\mathbf{x}) = \mathbf{H}^T(\mathbf{0})\mathbf{M}^{-1}(\mathbf{x})\mathbf{H}(\mathbf{x} - \mathbf{x}_I)\Phi_{a_I}(\mathbf{x} - \mathbf{x}_I) \quad (2.11)$$

The function $\Psi_I(\mathbf{x})$ is called the reproducing kernel approximation function or reproducing kernel shape function. Note that for $\mathbf{M}(\mathbf{x})$ to be non-singular, the support of $\Phi_{a_I}(\mathbf{x} - \mathbf{x}_I)$ has to cover enough discrete points so that the reproducing conditions are satisfied at any location in the domain. This is referred to as the kernel stability and that is dependent on the order of reproducing conditions imposed. Conditions assuring the non-singularity of the matrix $\mathbf{M}(\mathbf{x})$ as well as good performance of the method are discussed in [23], where rigorous convergence analysis and error estimates of the method are also provided.

3. Reproducing Kernel Interpolation

3.1 Modified Reproducing Kernel Approximation

The reproducing kernel shape functions discussed in Section 2 are in general not interpolation functions. This leads to complications in imposing essential boundary conditions and in applying nodal forces. It is therefore desirable to construct an interpolation function that preserves reproducing kernel features. For this purpose, a modified reproducing kernel approximation of $u(\mathbf{x})$ is considered as follows:

$$u^h(\mathbf{x}) = \sum_I \Psi_I(\mathbf{x}) u_I \quad (3.1)$$

$$\Psi_I(\mathbf{x}) = \hat{\Psi}_I(\mathbf{x}) + \bar{\Psi}_I(\mathbf{x}) \quad (3.2)$$

In Eq. (3.2), $\hat{\Psi}_I(\mathbf{x})$ is a primitive function used to introduce Kronecker delta properties in the approximation, and $\bar{\Psi}_I(\mathbf{x})$ is an enrichment function for imposing n-th order reproducing conditions:

$$\sum_I [\hat{\Psi}_I(\mathbf{x}) + \bar{\Psi}_I(\mathbf{x})] \mathbf{x}_I^\alpha = \mathbf{x}^\alpha; \quad |\alpha| \leq n \quad (3.3)$$

In the following discussion, we will show that if $\hat{\Psi}_I(\mathbf{x}_J) = \delta_{IJ}$, and Eq. (3.3) holds, then the vector of the enrichment functions, $\bar{\Psi}(\mathbf{x}) = \{\bar{\Psi}_1(\mathbf{x}), \bar{\Psi}_2(\mathbf{x}), \dots, \bar{\Psi}_{NP}(\mathbf{x})\}$, is orthogonal to the vector consisting of shifted monomial functions

$\mathbf{h}_i(\mathbf{x}) = \{h_i(\mathbf{x} - \mathbf{x}_1), h_i(\mathbf{x} - \mathbf{x}_2), \dots, h_i(\mathbf{x} - \mathbf{x}_{NP})\}$ where $h_i(\mathbf{x} - \mathbf{x}_I)$ is the i -th component of

$\mathbf{H}(\mathbf{x} - \mathbf{x}_I)$ in Eq. (2.2), at all discrete points $\{\mathbf{x}_J\}_{J=1}^{NP}$, i.e.,

$$\bar{\Psi}(\mathbf{x}_J)^T \mathbf{h}_i(\mathbf{x}_J) = 0, \quad \forall J \quad (3.4)$$

To show Eq. (3.4), Eq. (3.3) is first recast as

$$\sum_I [\hat{\Psi}_I(\mathbf{x}) + \bar{\Psi}_I(\mathbf{x})] (\mathbf{x} - \mathbf{x}_I)^\alpha = \delta_{|\alpha|,0}, \quad |\alpha| \leq n \quad (3.5)$$

or

$$\sum_I [\hat{\Psi}_I(\mathbf{x}) + \bar{\Psi}_I(\mathbf{x})] \mathbf{H}(\mathbf{x} - \mathbf{x}_I) = \mathbf{H}(\mathbf{0}) \quad (3.6)$$

Evaluating Eq. (3.6) at point \mathbf{x}_J leads to

$$\sum_I [\bar{\Psi}_I(\mathbf{x}_J) + \hat{\Psi}_I(\mathbf{x}_J)] \mathbf{H}(\mathbf{x}_J - \mathbf{x}_I) = \mathbf{H}(\mathbf{0}) \quad (3.7)$$

If $\hat{\Psi}_I(\mathbf{x}_J) = \delta_{IJ}$, one has

$$\sum_I [\bar{\Psi}_I(\mathbf{x}_J) + \delta_{IJ}] \mathbf{H}(\mathbf{x}_J - \mathbf{x}_I) = \mathbf{H}(\mathbf{0}) \quad (3.8)$$

Since $\sum_I \delta_{IJ} \mathbf{H}(\mathbf{x}_I - \mathbf{x}_J) = \mathbf{H}(\mathbf{0}) \quad \forall J$, Eq. (3.8) reduces to

$$\sum_I \bar{\Psi}_I(\mathbf{x}_J) \mathbf{H}(\mathbf{x}_J - \mathbf{x}_I) = \mathbf{0} \quad (3.9)$$

or equivalently,

$$\sum_I \bar{\Psi}_I(\mathbf{x}_J) h_i(\mathbf{x}_J - \mathbf{x}_I) = 0 \quad (3.10)$$

This completes the proof of Eq. (3.4). If indeed Eq. (3.10) leads to a trivial solution $\bar{\Psi}_I(\mathbf{x}_J) = 0$, then it follows that $\Psi_I(\mathbf{x})$ which satisfies reproducing condition in Eq. (3.3) becomes an interpolation function. This can be achieved by considering the following formulation for $\bar{\Psi}_I(\mathbf{x})$:

$$\bar{\Psi}_I(\mathbf{x}) = \mathbf{G}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{a}(\mathbf{x}) \quad (3.11)$$

where $\mathbf{G}(\mathbf{x} - \mathbf{x}_I)$ is the vector of basis functions that has the same dimension n as $\mathbf{H}(\mathbf{x} - \mathbf{x}_I)$, and $\mathbf{a}(\mathbf{x})$ is the corresponding coefficient vector. Substituting Eq. (3.11) into Eq. (3.9) leads to

$$\mathbf{Q}(\mathbf{x}_J) \mathbf{a}(\mathbf{x}_J) = \mathbf{0} \quad (3.12)$$

where

$$\mathbf{Q}(\mathbf{x}) = \sum_I \mathbf{H}(\mathbf{x} - \mathbf{x}_I) \mathbf{G}^T(\mathbf{x} - \mathbf{x}_I) \quad (3.13)$$

If $\mathbf{Q}(\mathbf{x}_J)$ is nonsingular, Eq. (3.13) yields a trivial solution $\mathbf{a}(\mathbf{x}_J) = \mathbf{0}$. It follows immediately $\bar{\Psi}_I(\mathbf{x}_J) = 0$ from Eq. (3.11), and thus

$$\Psi_I(\mathbf{x}_J) = \bar{\Psi}_I(\mathbf{x}_J) + \hat{\Psi}_I(\mathbf{x}_J) = \delta_{IJ} \quad (3.14)$$

One obvious choice of $\mathbf{G}(\mathbf{x} - \mathbf{x}_I)$ for nonsingular $\mathbf{Q}(\mathbf{x})$ is

$$\mathbf{G}(\mathbf{x} - \mathbf{x}_I) = \mathbf{H}(\mathbf{x} - \mathbf{x}_I) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \quad (3.15)$$

where $\bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I)$ is a positive function with compact support. Hence,

$$\bar{\Psi}_I(\mathbf{x}) = \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{a}(\mathbf{x}) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \quad (3.16)$$

Conclusion. If the enrichment function $\bar{\Psi}_I(\mathbf{x})$ takes the form of Eq. (3.11), and the primitive function $\hat{\Psi}_I(\mathbf{x})$ has Kronecker delta properties, satisfaction of reproducing conditions in Eq. (3.3) results in an interpolation function $\Psi_I(\mathbf{x})$.

The finite element shape functions is apparently a possible option for $\hat{\Psi}_I(\mathbf{x})$. This reduces to the approach of Huerta et al. [16]. In this paper, we consider the following simple construction of

$\hat{\Psi}_I(\mathbf{x})$:

$$\hat{\Psi}_I(\mathbf{x}) = \frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})}; \quad \hat{a}_I < \min\{\|\mathbf{x}_I - \mathbf{x}_J\|, \forall J \neq I\} \quad (3.17)$$

The support size \hat{a}_I of $\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)$ is so chosen that it does not cover any neighbor points, and thus Kronecker delta conditions are satisfied in $\hat{\Psi}_I(\mathbf{x})$.

3.2 Reproducing Conditions

Introducing Eqns. (3.16) and (3.17) into Eq. (3.2), we obtain the coefficients $\mathbf{a}(\mathbf{x})$ in $\bar{\Psi}_I(\mathbf{x})$ by the reproducing condition:

$$\sum_I \left[\frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})} + \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{a}(\mathbf{x}) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \right] \mathbf{x}_I^\alpha = \mathbf{x}^\alpha, \quad |\alpha| \leq n \quad (3.18)$$

Equation (3.18) can be rewritten as:

$$\sum_I \left[\frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})} + \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{a}(\mathbf{x}) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \right] (\mathbf{x} - \mathbf{x}_I)^\alpha = \delta_{|\alpha|,0}, \quad |\alpha| \leq n \quad (3.19)$$

or

$$\sum_I \mathbf{H}(\mathbf{x} - \mathbf{x}_I) \left[\frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})} + \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{a}(\mathbf{x}) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \right] = \mathbf{H}(\mathbf{0}) \quad (3.20)$$

The coefficient vector $\mathbf{a}(\mathbf{x})$ is obtained from Eq. (3.20) by

$$\mathbf{a}(\mathbf{x}) = \mathbf{Q}^{-1}(\mathbf{x})[\mathbf{H}(\mathbf{0}) - \hat{\mathbf{F}}(\mathbf{x})] \quad (3.21)$$

$$\hat{\mathbf{F}}(\mathbf{x}) = \sum_I \mathbf{H}(\mathbf{x} - \mathbf{x}_I) \frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})} \quad (3.22)$$

Finally the reproducing kernel interpolation function is obtained:

$$\Psi_I(\mathbf{x}) = \frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})} + \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{Q}^{-1}(\mathbf{x})[\mathbf{H}(\mathbf{0}) - \hat{\mathbf{F}}(\mathbf{x})] \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \quad (3.23)$$

It can be easily shown that

$$\begin{aligned} \Psi_I(\mathbf{x}_J) &= \frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x}_J - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})} + \mathbf{H}^T(\mathbf{x}_J - \mathbf{x}_I) \mathbf{M}^{-1}(\mathbf{x}_J)[\mathbf{H}(\mathbf{0}) - \hat{\mathbf{F}}(\mathbf{x}_J)] \bar{\Phi}_{\bar{a}_I}(\mathbf{x}_J - \mathbf{x}_I) \\ &= \delta_{IJ} + \mathbf{H}^T(\mathbf{x}_J - \mathbf{x}_I) \mathbf{M}^{-1}(\mathbf{x}_J)[\mathbf{H}(\mathbf{0}) - \hat{\mathbf{F}}(\mathbf{x}_J)] \bar{\Phi}_{\bar{a}_I}(\mathbf{x}_J - \mathbf{x}_I) \end{aligned} \quad (3.24)$$

Since

$$\hat{\mathbf{F}}(\mathbf{x}_J) = \sum_I \mathbf{H}(\mathbf{x}_J - \mathbf{x}_I) \frac{\hat{\Phi}_{\hat{a}_I}(\mathbf{x}_J - \mathbf{x}_I)}{\hat{\Phi}_{\hat{a}_I}(\mathbf{0})} = \mathbf{H}(\mathbf{0}) \quad (3.25)$$

Consequently, $\Psi_I(\mathbf{x}_J) = \delta_{IJ}$. Note that in Eqns. (3.24) and (3.25) the property

$\hat{a}_I < \min\{\|\mathbf{x}_I - \mathbf{x}_J\|, \forall J \neq I\}$ has been used.

For demonstration purposes, consider here the construction of a one-dimensional reproducing kernel interpolation functions of a set of uniformly distributed discrete points with nodal distance $\Delta x = 0.1$. The two kernel functions are selected as $\bar{\Phi}_{\bar{a}_I}(x - x_I) = \phi\left(\frac{x - x_I}{\bar{a}_I}\right)$ and $\hat{\Phi}_{\hat{a}_I}(x - x_I) = \phi\left(\frac{x - x_I}{\hat{a}_I}\right)$, where ϕ is the cubic B-spline function, and $\bar{a}_I = 2\Delta x$, and $\hat{a}_I = 0.9\Delta x$. The two component functions $\bar{\Psi}_I(\mathbf{x})$ and $\hat{\Psi}_I(\mathbf{x})$, and combined reproducing kernel interpolation function $\Psi_I(\mathbf{x})$ are shown in Fig. 1. For irregular discretization, \hat{a}_I should be carefully selected so that $Supp(\hat{\Phi}_{\hat{a}_I})$ does not cover any neighbor points. A reproducing kernel interpolation of 1-dimensional irregular discretization is shown in Fig. 2.

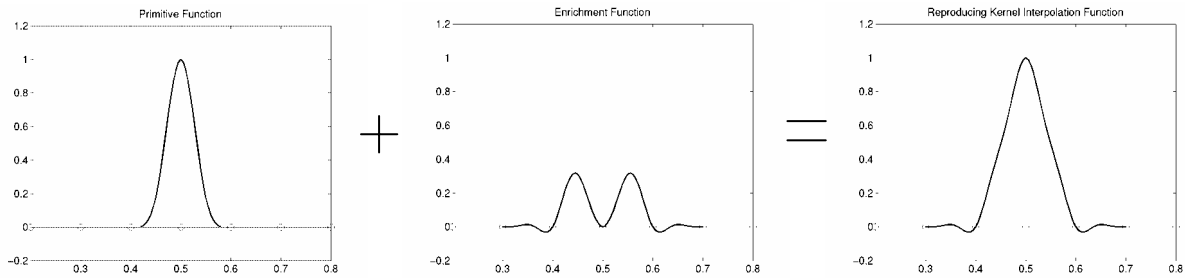


Figure 1 One-dimensional example of reproducing kernel interpolation function

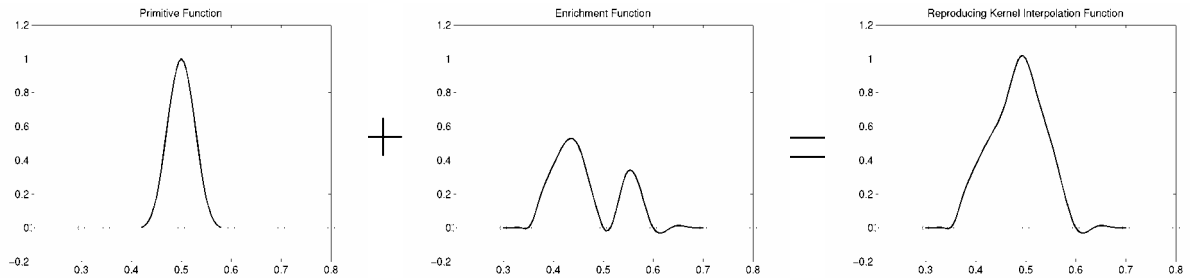


Figure 2 Reproducing kernel interpolation in one-dimensional irregular discretization

A representative discretization using reproducing kernel interpolation functions are shown in Fig.

3, where the supports with dash and solid lines refer to supports of $\hat{\Phi}_{\hat{a}_I}$ and $\bar{\Phi}_{\bar{a}_I}$, respectively.

Note that $\mathbf{x}_J \notin \text{Supp}(\hat{\Phi}_{\hat{a}_I}(\mathbf{x} - \mathbf{x}_I)), \forall J \neq I$.

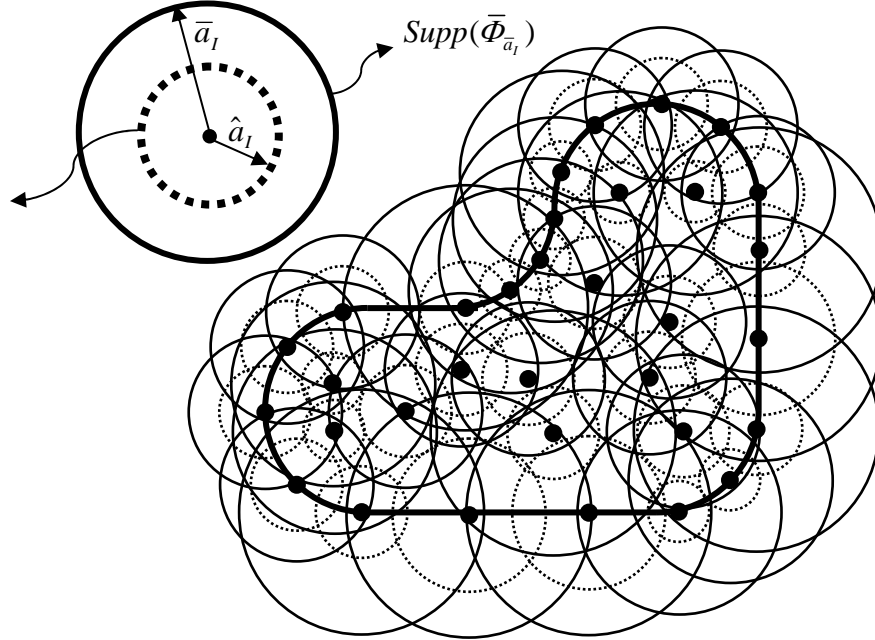


Figure 3 Domain discretization by reproducing kernel interpolation

The reproducing kernel interpolation function in Eq. (3.23) bares the following properties:

1. $\text{Supp}(\Psi_I(\mathbf{x})) = \max(\bar{a}_I, \hat{a}_I)$
2. If $\bar{\Phi}_{\bar{a}_I} \in C^{\bar{m}}$, $\hat{\Phi}_{\hat{a}_I} \in C^{\hat{m}}$, then $\Psi_I \in C^k$, $k = \min(\bar{m}, \hat{m})$
3. The singularity of $\mathcal{Q}(\mathbf{x})$ is only dependent on \bar{a}_I and the order of basis function in $\mathcal{G}(\mathbf{x} - \mathbf{x}_I)$, and is independent to \hat{a}_I .
4. It is flexible to include primitive functions only for the designated points. For example, let the primitive function $\hat{\Psi}_K(\mathbf{x})$ be only included for point \mathbf{x}_K , i.e., $\Psi_I(\mathbf{x}) = \bar{\Psi}_I(\mathbf{x})$ for

$l \neq K$, and $\Psi_K(\mathbf{x}) = \hat{\Psi}_K(\mathbf{x}) + \bar{\Psi}_K(\mathbf{x})$. Following Eqns. (3.23) – (3.25) it shows that $\Psi_K(\mathbf{x}_K) = 1$, and $\Psi_l(\mathbf{x}_K) = 0$ for all $l \neq K$. Note that $\Psi_K(\mathbf{x})$ does not vanish at other points, $\Psi_K(\mathbf{x}_l) \neq 0$. It is clear that in the approximation of $u(\mathbf{x})$ the nodal value of point x_K becomes the nodal value. For the purpose of a direct imposition of essential boundary conditions, the primitive functions can be added only to the essential boundary points. Figure 4 shows various conditions of including primitive functions in the construction of reproducing kernel approximation: (a) no primitive function is included (traditional reproducing kernel approximation), (b) primitive functions are added to all points, (c) primitive function is included only for a single point.

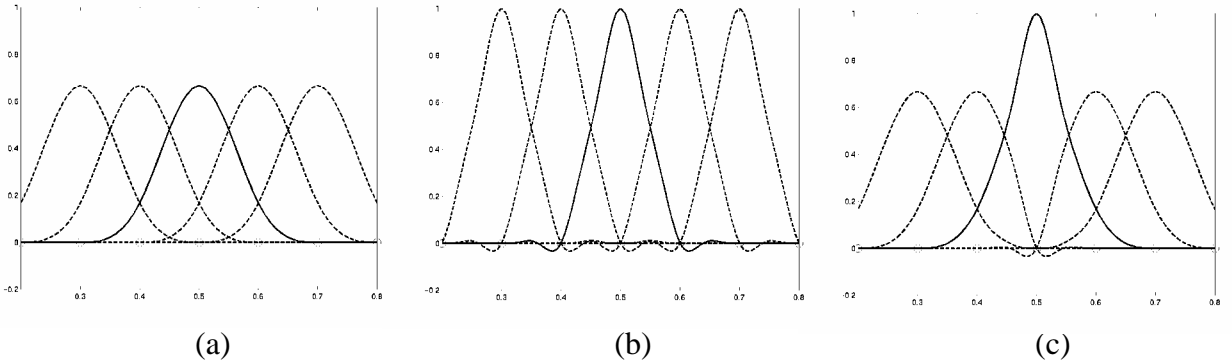


Figure 4 (a) Reproducing kernel approximation, (b) Reproducing kernel interpolation, (c) mixed interpolation at single point.

As mentioned earlier, employment of finite element shape function for $\Psi_l(\mathbf{x})$ reduces reproducing kernel interpolation to that of Huerta et al [16]. However, this approach requires a finite element mesh for construction of finite element shape functions. It is also noted that since piecewise linear finite element shape functions possess linear completeness, at least second order basis functions are needed in the construction of the enrichment function $\bar{\Psi}_l(\mathbf{x})$ in [16].

Otherwise, $\bar{\Psi}_I(\mathbf{x}) \equiv 0$ and $\Psi_I(\mathbf{x}) \equiv N_I(\mathbf{x})$. This requires the increase of support size of $\bar{\Phi}_{a_I}$ for nonsingular $\mathbf{Q}(\mathbf{x})$, and that consumes higher computational effort in the construction of stiffness matrix and force vector. Further, $\Psi_I \in C^0$. The formulation proposed in this paper resolves this dilemma. Since the primitive function $\hat{\Psi}_I(\mathbf{x})$ doesn't possess the linear completeness, enrichment function $\bar{\Psi}_I(\mathbf{x})$ with the low order consistency will not vanish like that in FEM enrichment. Hence, smaller support size can be used for $\bar{\Phi}_{a_I}(\mathbf{x} - \mathbf{x}_I)$ and arbitrary order of smoothness in Ψ_I can be obtained.

4. Error Analysis

For definiteness, we provide an error analysis of the proposed method when primitive functions are included only for the discrete points on the (relatively closed) part of the boundary $\Gamma_D \subset \partial\Omega$ where Dirichlet boundary conditions are specified. The case of including primitive functions for an arbitrarily chosen set of discrete points can be discussed with exactly the same technique. Let $\{\{\mathbf{x}_I\}_{I=1}^{NP}\} \subset \bar{\Omega}$ be a family of discrete point distributions. Thus a continuous function $u \in C(\bar{\Omega})$ is approximated by

$$u(\mathbf{x}) \approx \sum_I \Psi_I(\mathbf{x}) u(\mathbf{x}_I) \quad (4.1)$$

The meshfree shape function is

$$\Psi_I(\mathbf{x}) = \begin{cases} \bar{\Psi}_I(\mathbf{x}) & \text{for } \mathbf{x}_I \notin \Gamma_D \\ \bar{\Psi}_I(\mathbf{x}) + \hat{\Psi}_I(\mathbf{x}) & \text{for } \mathbf{x}_I \in \Gamma_D \end{cases} \quad (4.2)$$

where $\hat{\Psi}_I(\mathbf{x})$ are the primitive functions and $\bar{\Psi}_I(\mathbf{x})$ are the enrichment functions

$$\bar{\Psi}_I(\mathbf{x}) = \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{a}(\mathbf{x}) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \quad (4.3)$$

with the coefficient function vector $\mathbf{a}(\mathbf{x}) \in R^{N_n}$ determined by reproducing conditions. For any

$\mathbf{x}_I \in \Gamma_D$, the assumptions on $\hat{\Psi}_I$ are:

1. $\text{Supp}(\hat{\Psi}_I)$ does not contain \mathbf{x}_J for $J \neq I$;
2. $\hat{\Psi}_I(\mathbf{x}_I) = 1$

Then the Kronecker delta property holds for the particles on Γ_D :

$$\hat{\Psi}_I(\mathbf{x}_J) = \delta_{IJ}, \quad 1 \leq I \leq NP, \quad \mathbf{x}_J \in \Gamma_D \quad (4.4)$$

We will derive error estimates for the case of quasiuniform support sizes, i.e.

$$c_1 \leq \frac{\bar{a}_I}{a_J} \leq c_2 \quad \forall I, J \quad (4.5)$$

for two constants $0 < c_1 \leq c_2 < \infty$. Then there exists two constants $0 < \bar{c}_1 \leq \bar{c}_2 < \infty$ such that for any discrete point distribution $\{\mathbf{x}_I\}_{I=1}^{NP}$, we have a typical support size \bar{a} and

$$\bar{c}_1 \leq \frac{\bar{a}_I}{\bar{a}} \leq \bar{c}_2 \quad \forall I \quad (4.6)$$

Now the polynomial reproducing condition

$$\sum_I \Psi_I(\mathbf{x}) \mathbf{x}_I^\alpha = \mathbf{x}^\alpha, \quad \forall \alpha : |\alpha| \leq n \quad (4.7)$$

is equivalent to

$$\sum_I \Psi_I(\mathbf{x})(\mathbf{x} - \mathbf{x}_I)^\alpha = \delta_{|\alpha|,0}, \quad \forall \alpha : |\alpha| \leq n \quad (4.8)$$

This reproducing condition is rewritten as a linear system for the coefficient $a_\alpha(\mathbf{x})$:

$$[\sum_I \mathbf{H}(\mathbf{x} - \mathbf{x}_I) \mathbf{H}^T(\mathbf{x} - \mathbf{x}_I) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I)] \mathbf{a}(\mathbf{x}) = \mathbf{H}(\mathbf{0}) - \sum_{\mathbf{x}_I \in \Gamma_D} \hat{\Psi}_I(\mathbf{x}) \mathbf{H}(\mathbf{x} - \mathbf{x}_I) \quad (4.9)$$

Denote

$$\mathbf{M}_0(\mathbf{x}) = \sum_I \mathbf{H}\left(\frac{\mathbf{x} - \mathbf{x}_I}{\bar{a}}\right) \mathbf{H}^T\left(\frac{\mathbf{x} - \mathbf{x}_I}{\bar{a}}\right) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \quad (4.10)$$

for the scaled moment matrix, and $\tilde{\mathbf{a}}(\mathbf{x}) = (\bar{a}^{|\alpha|} a_\alpha(\mathbf{x}))_{|\alpha| \leq n}$. Then the polynomial reproducing condition implies

$$\mathbf{M}_0(\mathbf{x}) \tilde{\mathbf{a}}(\mathbf{x}) = \mathbf{H}(\mathbf{0}) - \sum_{\mathbf{x}_I \in \Gamma_D} \hat{\Psi}_I(\mathbf{x}) \mathbf{H}\left(\frac{\mathbf{x} - \mathbf{x}_I}{\bar{a}}\right) \quad (4.11)$$

To describe conditions under which the method is defined and works well, we bring in some definitions. We say that a point $x \in \bar{\Omega}$ is covered by m shape functions if there are m indices I_1, \dots, I_m such that

$$\|\mathbf{x} - \mathbf{x}_{I_j}\| < \bar{a}_{I_j}, \quad j = 1, \dots, m \quad (4.12)$$

We say a family of discrete point distributions $\{\{\mathbf{x}_I\}_{I=1}^{NP}\}$ is (\bar{a}, n) -regular if there is a constant L_0 such that

$$\max_{\mathbf{x} \in \bar{\Omega}} \|\mathbf{M}_0(\mathbf{x})\|_2 \leq L_0 \quad (4.13)$$

for all the discrete point distributions in the family.

It can be shown that a family of one-dimensional discrete point distributions is (\bar{a}, n) -regular if any point is covered by at least $n+1$ shape functions and in any local region there are at least $n+1$ discrete points with relative distances bounded away from 0. For a general d -dimensional domain, a family of discrete point distributions is (\bar{a}, l) -regular if any point is covered by $d+l$ shape functions and the d -simplex with the vertices of the corresponding discrete points has a volume larger than $\bar{c}_0 \bar{a}^d$. For details, see [23].

In Eq. (3.17), a concrete form for the primitive functions is suggested. For error analysis of the method, we make the following assumptions.

$$\left. \begin{aligned} 1. \text{Supp}(\hat{\Psi}_I) = B(\mathbf{x}; \hat{a}_I) \text{ does not contain } \mathbf{x}_J \text{ for } J \neq I. \\ 2. \text{There are constrains } 0 < \hat{c}_1 \leq \hat{c}_2 < \infty \text{ such that } \hat{c}_1 \bar{a} \leq \hat{a}_I \leq \hat{c}_2 \bar{a} \text{ for any } \mathbf{x}_I \in \Gamma_D \\ 3. \hat{\Psi}_I(\mathbf{x}_I) = 1 \\ 4. \hat{\Psi}_I \in C^k \end{aligned} \right\} (4.14)$$

By a standard scaling argument, we have the bounds

$$\max_{|\beta|=l} \|D^\beta \hat{\Psi}_I\|_{L^\infty(\Omega)} \leq c\bar{a}^{-l}, \quad l=0,1,\dots,k, \quad \forall I \quad (4.15)$$

Notice that in the particular case of uniform discrete point distributions and uniform support sizes, we may simply take

$$\hat{\Psi}_I(\mathbf{x}) = \hat{\Psi}\left(\frac{\mathbf{x} - \mathbf{x}_I}{\bar{a}}\right) \quad (4.16)$$

for some (fixed) function $\hat{\Psi}$ with the properties

$$\left. \begin{aligned} 1. \text{ Supp}(\hat{\Psi}_I) = B(\mathbf{0}; c_0) \text{ is the ball centered at the origin with radius } c_0 \text{ for} \\ \text{any } c_0 \in (0,1); \\ 2. \hat{\Psi}(\mathbf{0}) = 1; \\ 3. \hat{\Psi} \in C^k(B(\mathbf{0}; c_0)) \end{aligned} \right\} \quad (4.17)$$

It is easy to see that the requirement Eq. (4.14) on the functions $\hat{\Psi}_I$ are all satisfied with

$$\hat{c}_1 = \hat{c}_2 = 1.$$

In the following we always assume the regularity of the discrete point distributions. We also

assume $\bar{\Phi} \in C^k$. From Eq. (4.11) we immediately obtain

$$\max_{\alpha:|\alpha|\leq n} \bar{a}^{|\alpha|} \|a_\alpha\|_{L^\infty(\Omega)} \leq c \quad (4.18)$$

for some constant $c < \infty$.

Rewrite the enrichment functions (4.3) in the form

$$\bar{\Psi}_I(\mathbf{x}) = \mathbf{H}^T \left(\frac{\mathbf{x} - \mathbf{x}_I}{\bar{a}} \right) \bar{\Phi}_{\bar{a}_I}(\mathbf{x} - \mathbf{x}_I) \bar{a}^{|\alpha|} a_\alpha(\mathbf{x}) \quad (4.19)$$

Together with the bound Eq. (4.15), we see that the meshfree shape functions are uniformly bounded:

$$\max_{1 \leq I \leq NP} \|\Psi_I\|_{L^\infty(\Omega)} \leq c \quad (4.20)$$

We then bound the first derivatives of the shape functions. For any multi-index β with $|\beta| = 1$,

we differentiate the system Eq. (4.11) to obtain

$$\begin{aligned} \mathbf{M}_0(\mathbf{x}) D^\beta \tilde{\mathbf{a}}(\mathbf{x}) &= D^\beta \mathbf{M}_0(\mathbf{x}) \tilde{\mathbf{a}}(\mathbf{x}) \\ &\quad - \sum_{x_I \in \Gamma_D} [D^\beta \hat{\Psi}_I(\mathbf{x}) \mathbf{H}^T \left(\frac{\mathbf{x} - \mathbf{x}_I}{\bar{a}} \right) + \hat{\Psi}_I(\mathbf{x}) D^\beta \mathbf{H}^T \left(\frac{\mathbf{x} - \mathbf{x}_I}{\bar{a}} \right)] \end{aligned} \quad (4.21)$$

Using Eq. (4.15), we know that the summation term on the right hand side of Eq. (4.21) is bounded by $c\bar{a}^{-l}$. Also, easily,

$$\max_{\beta:|\beta|=l} \max_{\mathbf{x} \in \bar{\Omega}} \left\| D^\beta \mathbf{M}_0(\mathbf{x}) \right\|_2 \leq c\bar{a}^{-l} \quad (4.22)$$

Thus from Eq. (4.21), we have

$$\max_{\beta:|\beta|=l} \max_{\alpha:|\alpha|\leq n} \bar{a}^{|\alpha|} \left\| D^\beta a_\alpha \right\|_{L^\infty(\Omega)} \leq c\bar{a}^{-l} \quad (4.23)$$

In general, an inductive argument shows that

$$\max_{\alpha:|\alpha|\leq n} \max_{\beta:|\beta|=l} \bar{a}^{|\alpha|} \left\| D^\beta a_\alpha \right\|_{L^\infty(\Omega)} \leq c\bar{a}^{-l}, l=0,1,\dots,k \quad (4.24)$$

Then from Eq. (4.19), we obtain the following result.

Theorem 4.1 Assume the discrete point distributions are (\bar{a}, n) -regular, $\bar{\Phi}$ is k-times continuously differentiable, and Eq. (4.14) holds. Then

$$\max_I \max_{\beta:|\beta|=l} \left\| D^\beta \Psi_I \right\|_{L^\infty(\Omega)} \leq c\bar{a}^{-l}, \quad 0 \leq l \leq k \quad (4.25)$$

Now we are ready to derive error estimates for the meshfree interpolants. For this purpose, we first introduce some definitions. Let $B \subset R^d$ be a ball. Then a domain Ω_I is said to be star-shaped with respect to B if for any $\mathbf{x} \in \Omega_I$, the closed convex hull of $\{\mathbf{x}\} \cup B$ is a subset of Ω_I .

The chunkiness parameter of Ω_I is defined to be Ω_I / ρ_{max} , where

$$\rho_{max} = \sup\{ \rho : \Omega_I \text{ is star-shaped with respect to a ball of radius } \rho \} \quad (4.26)$$

Let $u \in W^{m+1,q}(\Omega)$, $m \geq 0$, $q \in [1, \infty]$. We assume $(m+1)q > d$ if $q > 1$, or $m+1 \geq d$ if $q = 1$. Then by the Sobolev embedding theorem, $u \in C(\bar{\Omega})$, and it is meaningful to use pointwise values of $u(\mathbf{x})$. The meshfree interpolant of $u(\mathbf{x})$ is

$$u^I(\mathbf{x}) = \sum_I u(\mathbf{x}_I) \Psi_I(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega} \quad (4.27)$$

Notice that this function interpolates u at the discrete points on the Dirichlet boundary:

$$u^I(\mathbf{x}_I) = u(\mathbf{x}_I) \text{ for } \mathbf{x}_I \in \Gamma_D. \text{ Denote } n_I = \min\{m+1, n+1\}.$$

We will assume the following hypothesis is satisfied.

Hypothesis (H). *There is a constant integer N_0 such that for any $\mathbf{x} \in \bar{\Omega}$, there are at most N_0 of \mathbf{x}_I satisfying the relation $\|\mathbf{x} - \mathbf{x}_I\| < \bar{a}_I$, i.e., each point in $\bar{\Omega}$ is covered by at most N_0 shape functions.*

The hypothesis (H) is quite natural since otherwise as the number of shape functions covering a local area increases, the shape functions tend to be more and more linearly dependent in the local area.

To simplify the notation, we write $B_I \equiv B_{\bar{a}_I}(\mathbf{x}_I)$, $1 \leq I \leq NP$. We first bound the error $u - u^I$ in Sobolev norms over $B_J \cap \bar{\Omega}$ for $J = 1, \dots, NP$. Define

$$\Omega_J = \{ \mathbf{x} : \|\mathbf{x} - \mathbf{x}_J\| < \bar{a}_J + \max_{1 \leq I \leq NP} \bar{a}_I \} \quad (4.28)$$

and let

$$S_J = \{ I : \text{dist}(\mathbf{x}_I, B_J) < \bar{a}_I \} \quad (4.29)$$

It follows from the hypothesis (H) that $\text{card}(S_J)$, $1 \leq J \leq NP$, are uniformly bounded. If $\bar{\Omega}_J \subset \bar{\Omega}$, then $\bar{\Omega}_J \cap \bar{\Omega} = \bar{\Omega}_J$ is star-shaped with respect to $\tilde{B}_J \equiv B_J$, and the chunkiness parameter of $\bar{\Omega}_J \cap \bar{\Omega}$ is uniformly bounded. Now suppose $\bar{\Omega}_J \not\subset \bar{\Omega}$. Then since $\partial\Omega$ is Lipschitz continuous, if \bar{a} is sufficiently small, we can choose a ball \tilde{B}_J of radius $\bar{a}_J/2$ with \mathbf{x}_J on its boundary such that $\bar{\Omega}_J \cap \bar{\Omega}$ is star-shaped with respect to \tilde{B}_J . Again we see that the chunkiness parameter of $\bar{\Omega}_J \cap \bar{\Omega}$ is uniformly bounded.

Let $P_J^{n_l}u$ be the Taylor polynomial of degree $n_l - l$ of u averaged over \tilde{B}_J ([24, Section 4.1]), and denote

$$R_J^{n_l}u(\mathbf{x}) = u(\mathbf{x}) - P_J^{n_l}u(\mathbf{x}) \quad (4.30)$$

for the remainders. Then since the chunkiness parameters of $\bar{\Omega}_J \cap \bar{\Omega}$ are uniformly bounded when \bar{a} is sufficiently small, from the results of [24, Section 4.3], we have the estimates:

$$\|R_J^{n_l}u\|_{W^{l,q}(\Omega_J \cap \Omega)} \leq c\bar{a}^{n_l-l} \|u\|_{W^{n_l,q}(\Omega_J \cap \Omega)}, \quad l = 0, \dots, n_l \quad (4.31)$$

$$\|R_J^{n_l}u\|_{L^\infty(\Omega_J \cap \Omega)} \leq c\bar{a}^{n_l-d/q} \|u\|_{W^{n_l,q}(\Omega_J \cap \Omega)} \quad (4.32)$$

where the constant c depends only on n_l , d and q , and is independent of J .

For $x \in B_J \cap \bar{\Omega}$, we have

$$\begin{aligned} u(\mathbf{x}) - u^I(\mathbf{x}) &= P_J^{n_l}u(\mathbf{x}) - \sum_I P_J^{n_l}u(\mathbf{x}_I) \mathcal{P}_I(\mathbf{x}) \\ &\quad + R_J^{n_l}u(\mathbf{x}) - \sum_{I \in S_J} R_J^{n_l}u(\mathbf{x}_I) \mathcal{P}_I(\mathbf{x}) \end{aligned} \quad (4.33)$$

By the polynomial reproducing property,

$$\sum_I P_J^{n_I} u(\mathbf{x}_I) \Psi_I(\mathbf{x}) = P_J^{n_I} u(\mathbf{x}) \quad (4.34)$$

Thus we have the error representation formula

$$u(\mathbf{x}) - u^I(\mathbf{x}) = R_J^{n_I} u(\mathbf{x}) - \sum_{I \in S_J} R_J^{n_I} u(\mathbf{x}_I) \Psi_I(\mathbf{x}) \quad (4.35)$$

Note that $\mathbf{x}_I \in \bar{\Omega}_J \cap \bar{\Omega}$ for $I \in S_J$. So

$$\|u - u^I\|_{W^{l,q}(B_J \cap \Omega)} \leq \|R_J^{n_I} u\|_{W^{l,q}(B_J \cap \Omega)} + \|R_J^{n_I} u\|_{L^\infty(\Omega_J \cap \Omega)} \sum_{I \in S_J} \|\Psi_I\|_{W^{l,q}(B_J \cap \Omega)} \quad (4.36)$$

Since $\text{card}(S_J)$ is uniformly bounded, applying the estimates (4.31), (4.32) and Theorem 4.1, we have

$$\|u - u^I\|_{W^{l,q}(B_J \cap \Omega)} \leq c \bar{a}^{-n_I - l} |u|_{W^{n_I, q}(\Omega_J \cap \Omega)}, \quad 0 \leq l \leq \min\{n_I, k\}, \quad I \in S_J \leq NP \quad (4.37)$$

Therefore, recalling again the hypothesis (H),

$$\|u - u^I\|_{W^{l,q}(\Omega)} \leq c \bar{a}^{-n_I - l} |u|_{W^{n_I, q}(\Omega)}, \quad 0 \leq l \leq \min\{n_I, k\} \quad (4.38)$$

To conclude, we have derive the following optimal order meshfree interpolation

error estimates.

Theorem 4.2 *Assume the conditions stated in Theorem 4.1. Also assume the hypothesis (H).*

Let $m \geq 0$, $q \in [1, \infty]$ with $(m+1)q > d$ if $q > 1$, or $m+1 \geq d$ if $q = 1$. Then for any

$u \in W^{m+1,q}(\Omega)$, we have the optimal order interpolation error estimates

$$\|u - u^I\|_{W^{l,q}(\Omega)} \leq c\bar{a}^{\min\{m+1, n+1\}-l} |u|_{W^{\min\{m+1, n+1\}, q}(\Omega)}, \quad \forall l \leq \min\{m+1, n+1, k\} \quad (4.39)$$

Notice that when u is sufficiently smooth, $m \geq n$, the error estimate Eq. (4.39) reduces to

$$\|u - u^I\|_{W^{l,q}(\Omega)} \leq c\bar{a}^{n+1-l} |u|_{W^{n+1,q}(\Omega)}, \quad \forall l \leq \min\{n+1, k\} \quad (4.40)$$

If we further assume $\bar{\Phi}$ and $\hat{\Psi}_I$ are sufficiently smooth so that $k \geq n+1$, then

$$\|u - u^I\|_{W^{l,q}(\Omega)} \leq c\bar{a}^{n+1-l} |u|_{W^{n+1,q}(\Omega)}, \quad l = 0, 1, \dots, n+1 \quad (4.41)$$

For meshfree solutions of one-dimensional elliptic boundary value problems, it then follows that the optimal order error estimates hold:

$$\|u - u^R\|_{H^1(\Omega)} \leq c\bar{a}^n |u|_{H^{n+1}(\Omega)} \quad (4.42)$$

5. Convergence Study

In this section, we present some results on numerical experiments of the meshfree methods to demonstrate the theoretical error estimates derived in the previous section. Numerical examples showing the performance of the proposed method will be given in the next section.

For all the following numerical experiments, we use uniform discrete point distributions and uniform support sizes. Then the support size \bar{a} is proportional to the nodal distance h . From the results of the previous section, we have the following error estimates for the meshfree interpolant

u^I :

$$\|u - u^I\|_{H^1(\Omega)} \leq ch^n |u|_{H^{n+1}(\Omega)} \quad (5.1)$$

$$\|u - u^I\|_{L^2(\Omega)} \leq ch^{n+1} |u|_{H^{n+1}(\Omega)} \quad (5.2)$$

as long as the function u belongs to the space $H^{n+1}(\Omega)$. For a one-dimensional linear elliptic boundary value problem, we also have the optimal order meshfree error estimate of meshfree solution u^R :

$$\|u - u^R\|_{H^1(\Omega)} \leq ch^n |u|_{H^{n+1}(\Omega)} \quad (5.3)$$

By the well-known Nitsche's technique, we also have the optimal order meshfree error estimate in L^2 norm:

$$\|u - u^R\|_{L^2(\Omega)} \leq ch^{n+1} \|u\|_{H^{n+1}(\Omega)} \quad (5.4)$$

Two examples are analyzed to verify the order of convergence. The first example is a one-dimensional boundary value problem :

$$u_{,xx} = e^x \quad x \in (0,1) \quad (5.5)$$

$$u(0) = 1, \quad u(1) = e$$

The exact solution is $u(x) = e^x$. Uniform particle distributions and uniform support sizes are used for meshfree computation. Denote h the distance between two neighboring particles and n the polynomial degree. The support size of the enrichment functions is chosen as $\bar{a}/h = 2.1 + n$, and that of the primitive functions $\hat{a}/h = 0.5$. Figure 5 shows errors of reproducing kernel interpolations and Figure 6 the errors of meshfree solutions in L^2 related norms. Essential boundary conditions are imposed using the Kronecker delta property of reproducing kernel interpolation on the essential boundaries. We observe the optimal order convergence for both the meshfree interpolants and the meshfree solutions. By comparison of numerical solution using $\hat{a}/h = 0.1, 0.5, 0.9$, the results show that the support size of primitive function \hat{a} has very minimum influence on the solution accuracy and the order of convergence.

For the two-dimensional boundary value problem, we solve the following Poisson equation

$$\nabla^2 u(x, y) = (x^2 + y^2)e^{xy} \quad (x, y) \in (0, 1) \times (0, 1) \quad (5.6)$$

$$u(0, y) = 1, \quad u(1, y) = e^y, \quad 0 \leq y \leq 1$$

$$u(x, 0) = 1, \quad u(x, 1) = e^x, \quad 0 \leq x \leq 1$$

The exact solution is $u(x, y) = e^{xy}$. Uniform particle distribution and uniform support sizes are used for meshfree computation. Denote h the distance in the x or y direction between two neighboring particles and n the monomial degree. The support size of the enrichment functions is chosen as $\bar{a}/h = 2.1 + n$, and that of the primitive functions $\hat{a}/h = 0.5$. Numerical results shown in Fig. 7 indicate that we have optimal order convergence for the reproducing kernel interpolation. Meshfree solution using first order monomials shown in Fig. 8 also achieves optimum order of convergence. For higher monomial degrees, although the convergence order in the energy norm is still one, the corresponding meshfree solutions are more accurate than that with linear basis functions. This is due to the fact that the essential boundary conditions are imposed only at discrete points. Similar phenomenon was observed in the meshfree solution using standard reproducing kernel approximation.

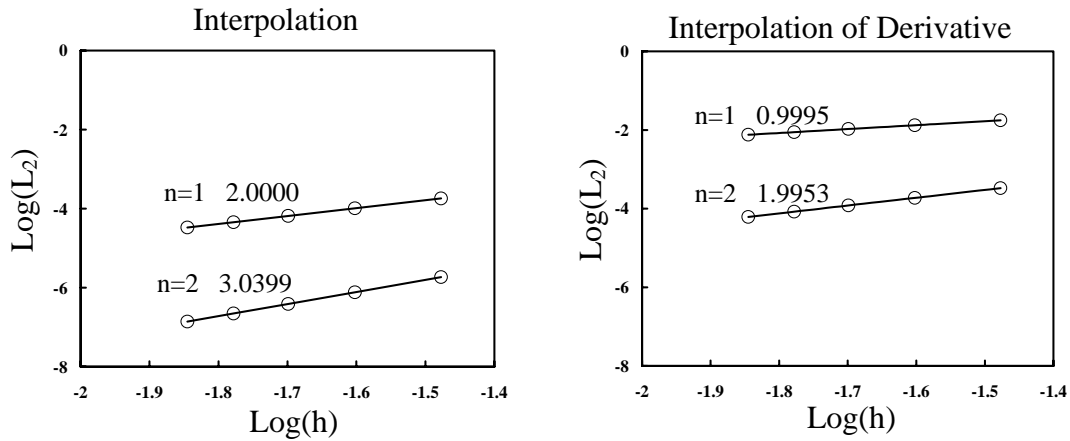


Figure 5 Error norms of one-dimensional reproducing kernel interpolation

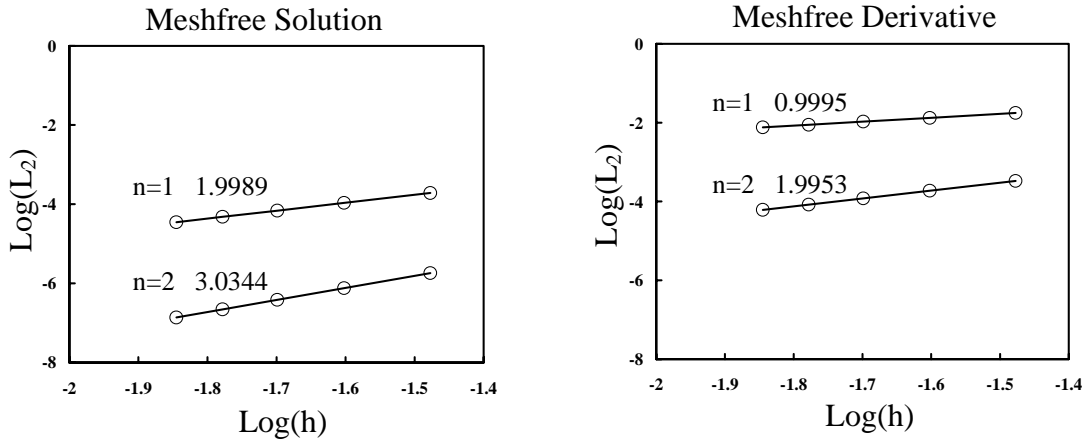


Figure 6 Error norms of one-dimensional meshfree solution

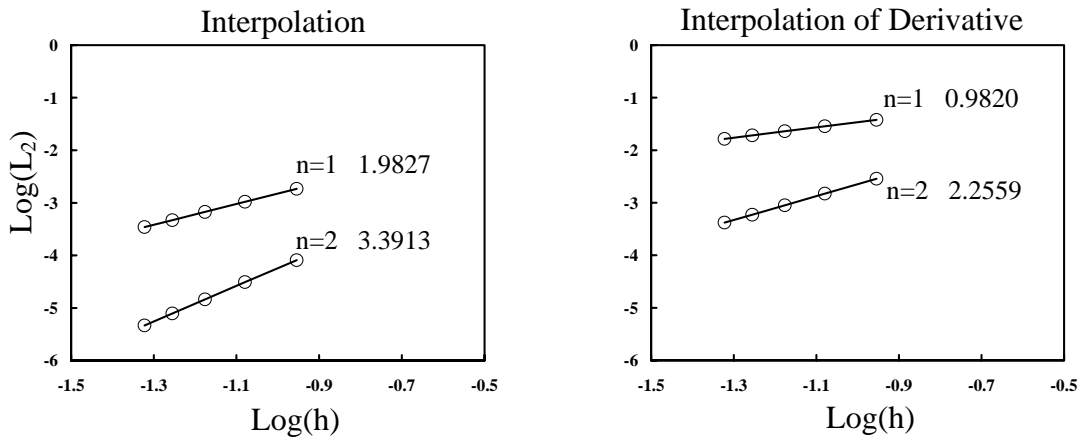


Figure 7 Error norms of two-dimensional reproducing kernel interpolation

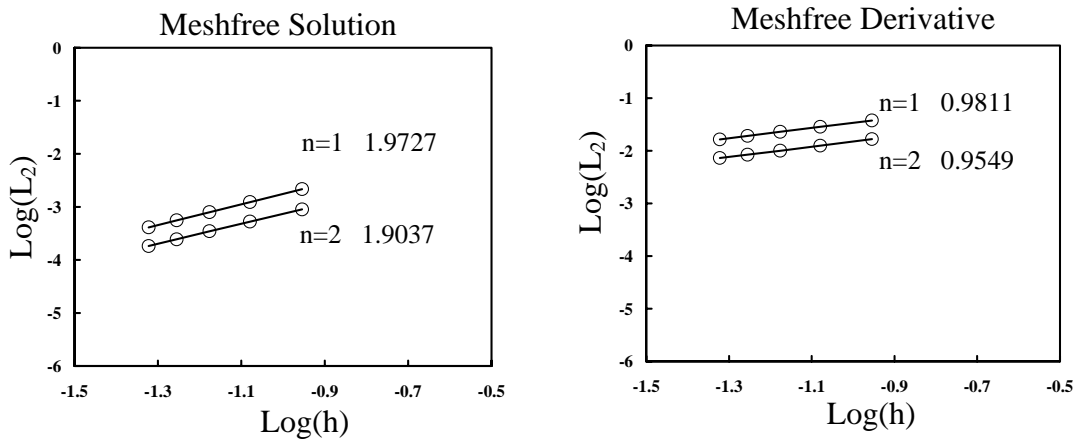


Figure 8 Error norms of two-dimensional meshfree solution

6. Numerical Examples

In the following examples, unless otherwise specified, mixed reproducing kernel interpolation (Mixed RKI) functions using cubic B-spline kernel with linear basis are employed. Normalized support sizes for $\bar{\Phi}_{\bar{a}_I}$ and $\hat{\Phi}_{\hat{a}_I}$ are defined as $\bar{R}_I = \bar{a}_I / h$ and $\hat{R}_I = \hat{a}_I / h$, where h is the nodal distance.

6.1 One-Dimensional High Order Equation

In this example, the following model problem is considered:

$$\begin{aligned} -u_{,xx} + 20u(x) &= 0 \quad \text{in } (0,1) \\ u(0) &= c_1, u(1) = c_1 \exp(\sqrt{20}) + c_2 \exp(-\sqrt{20}) \end{aligned} \quad (6.1)$$

where $c_1 = 1/(\exp(\sqrt{20}) - \exp(-\sqrt{20}))$, $c_2 = c_1 - 1/\sqrt{20}$, Reproducing kernel interpolation functions are constructed using linear basis functions with normalized support size $\hat{R}_I = 0.9$ for $\hat{\Phi}_{\hat{a}_I}$ and $\bar{R}_I = 2.0$ for $\bar{\Phi}_{\bar{a}_I}$. The solution is also compared with the regular reproducing kernel approximation with boundary conditions imposed by Lagrange multiplier method. The comparison of u and du/dx with an equally spaced 11-node model is shown in Fig. 9.

The convergence of the proposed formulation with respect to the support size of the interpolation function is also studied. For this purpose $\bar{R}_I = 1.2, 2.0, 3.0$ for $\bar{\Phi}_{\bar{a}_I}$ are considered in the analysis. Consecutively refined regular models with 11, 21, 41, and 81 points are used in the

convergence study. The error norms in u and du/dx are plotted in Fig. 10. The results show a characteristic similar to that of the conventional reproducing kernel approximation.

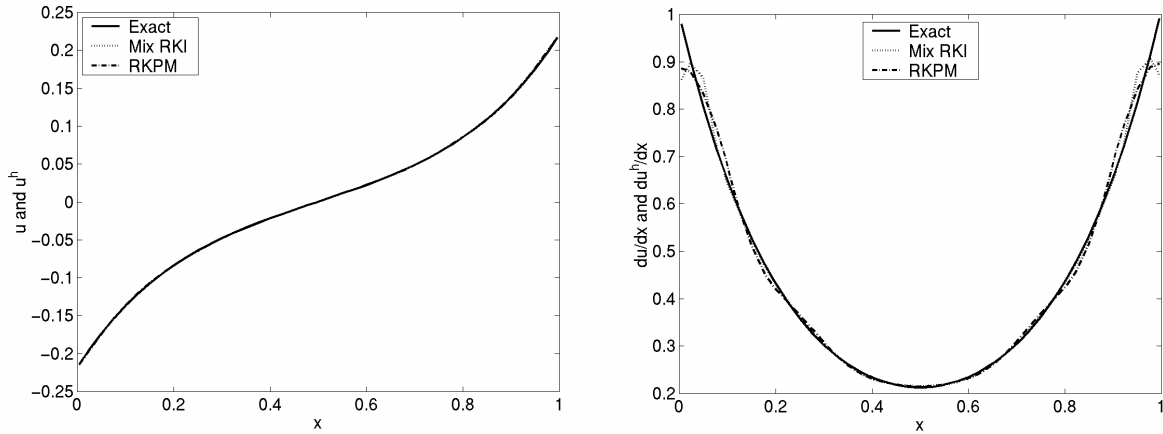


Figure 9 Solution comparison using 11 nodes regular model

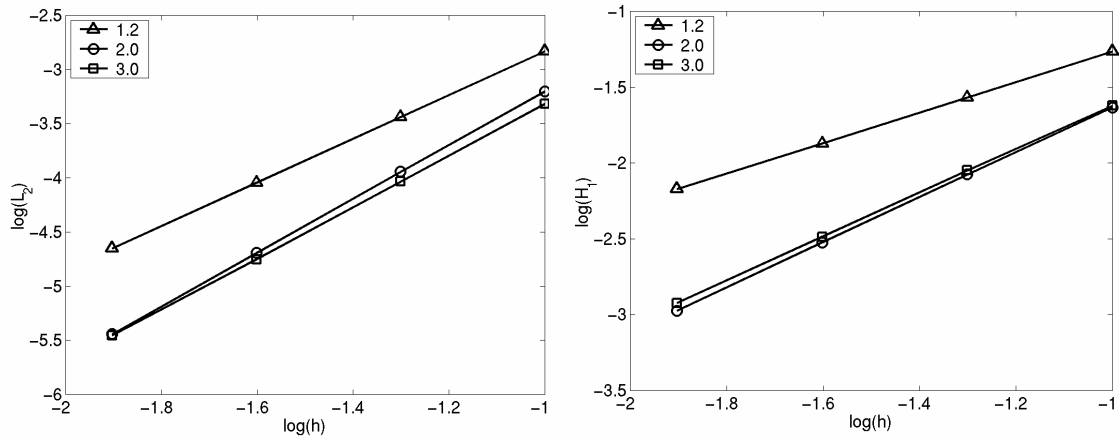
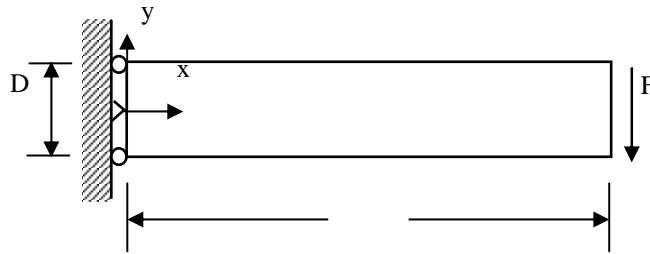


Figure 10 Error norms in u and du/dx for $\bar{R}_1 = 1.2, 2.0, 3.0$

6.2 Beam Bending

A cantilever beam subjected to a shear load as shown in Fig. 11 is studied. Due to the symmetry in the geometry, only half of the model is modeled. Three regular models with 11×3 , 21×5 , and 41×9 points, as shown in Fig. 12, are tested. Linear basis functions with normalized support sizes $\hat{R}_I = 0.9$ for $\hat{\Phi}_{a_I}$ and $\bar{R}_I = 2.0$ for $\bar{\Phi}_{a_I}$ are employed. The comparison of mixed reproducing kernel interpolation, the regular reproducing kernel approximation, and the finite element solution are shown in Table 1. It is noted that with the mixed reproducing kernel interpolation, the imposition of essential boundary conditions is much less costly than that of the regular reproducing kernel approximation with Lagrange multiplier method.



$$F=20\text{kN}, L=10.0 \text{ m}, D=2.0 \text{ m } E=21.1 \text{ MPa}, \nu= 0.3$$

Figure 11 Beam bending problem configuration

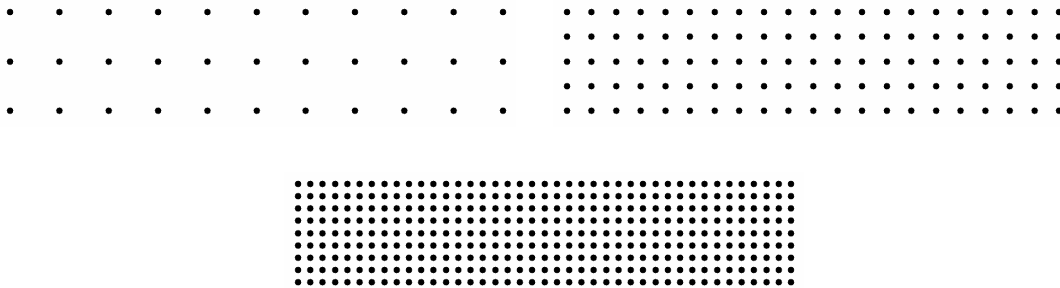


Figure 12 Regular models with 11×3 , 21×5 , and 41×9 nodes

Table 1 Tip displacement comparison in beam problem

Model	33	105	369
FEM	71.1956%	88.3096%	96.9314%
RKPM	74.3174%	93.5920%	98.4111
Mix RKI	74.0020%	93.6215%	98.4354

The displacement error norms of the proposed method using $\bar{R}_I = 1.2, 2.0, 3.0$ for $\bar{\Phi}_{\bar{a}_I}$ are shown in Fig. 14, and the corresponding vertical displacement along $y=0.0$ and shear stress distributions along $x=5.0$ using finest discretization are plotted in Figs. 14 and 15.

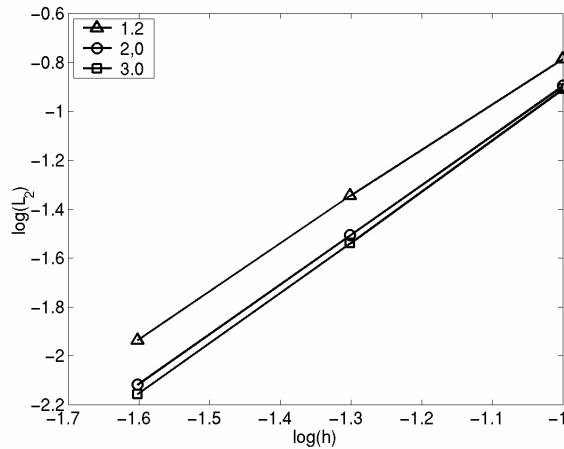


Figure 13 L_2 norm using $\bar{R}_I = 1.2, 2.0, 3.0$

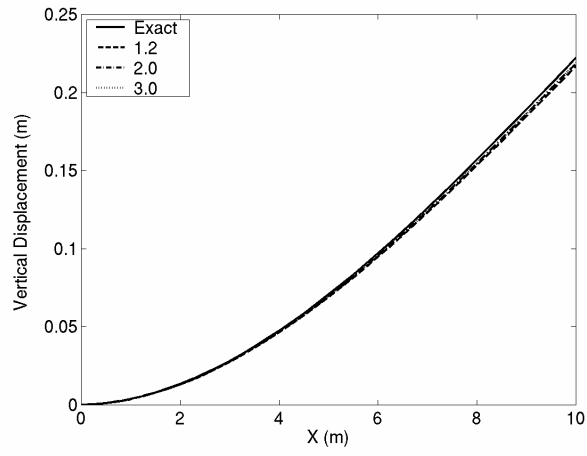


Figure14 Vertical displacement $y = 0.0$ using $\bar{R}_I = 1.2, 2.0, 3.0$

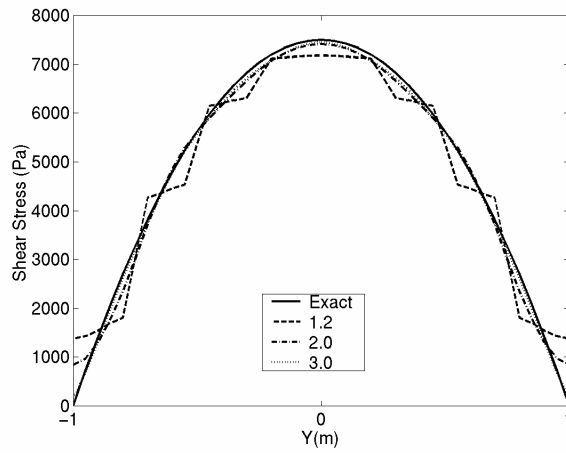


Figure15 Shear stress along $x = 5.0$ using $\bar{R}_I = 1.2, 2.0, 3.0$

6.3 Two Dimension Plate with Hole

An infinite plate with a central circular hole (diameter $2a$) subjected to a unidirectional tensile load 1.0 in the x direction as show in Fig. 16 is studied using the proposed method. Plane strain condition is assumed with Young's modulus $E = 2.11 \times 10^{11} Pa$ and Poisson's ratio $\nu = 0.3$. The exact solution of this problem is given in the polar coordinates:

$$\begin{aligned}\sigma_{xx} &= 1 - \frac{a^2}{r^2} \left(\frac{3}{2} \cos(2\phi) + \cos(4\phi) \right) + \frac{3a^4}{2r^4} \cos(4\phi) \\ \sigma_{yy} &= -\frac{a^2}{r^2} \left(\frac{1}{2} \cos(2\phi) - \cos(4\phi) \right) - \frac{3a^4}{2r^4} \cos(4\phi) \\ \tau_{xy} &= -\frac{a^2}{r^2} \left(\frac{1}{2} \sin(2\phi) + \sin(4\phi) \right) + \frac{3a^4}{2r^4} \sin(4\phi)\end{aligned}\tag{6.3}$$

where (r, ϕ) are the usual polar co-ordinates. Due to geometric symmetry, only half model was modeled by three domain discretizations using 45, 153, and 561 points as shown in Fig. 17. The traction along the edges $x = \pm 5$ and $y = 5$ are given according to the analytic solution Eq. (6.3)

The normal stresses σ_{xx} along the line $x = 5$ using the mixed reproducing kernel interpolation (Mixed RKI) with three discretizations are presented in Fig. 18. Results comparison using the Mixed RKI and the conventional reproducing kernel approximation with the finest discretization is shown in Fig. 19, and the corresponding energy norms are compared in Fig 20.

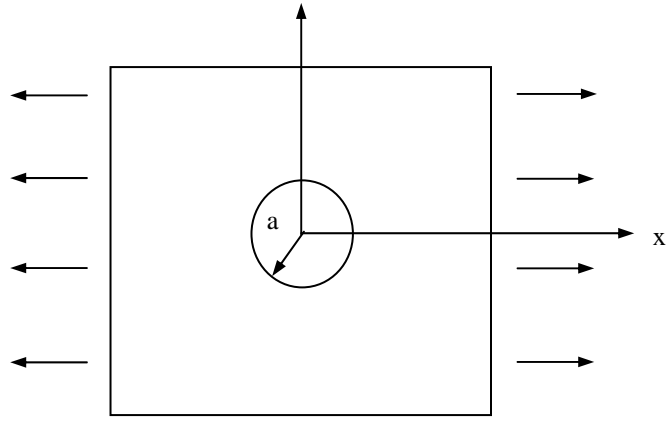


Figure 16 Problem statement

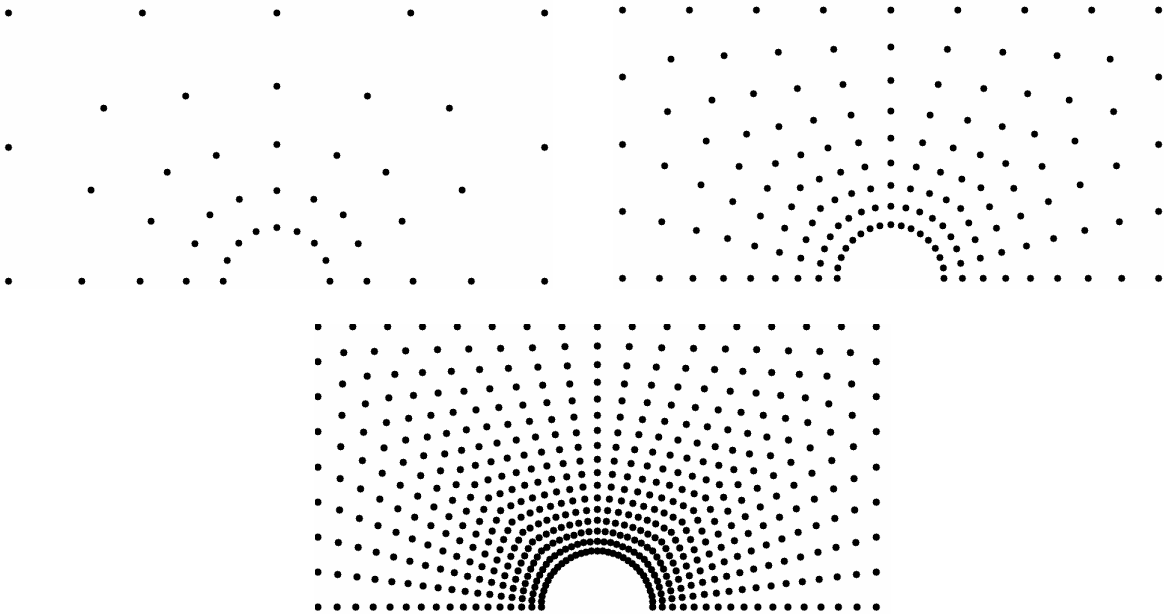


Figure17 Domain discretization with 45, 153 and 561 Nodes

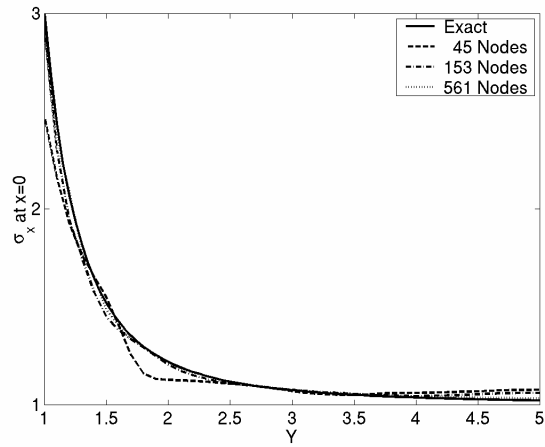


Figure 18 Stress plot along $x=0$ for 3 models using Mix RKI

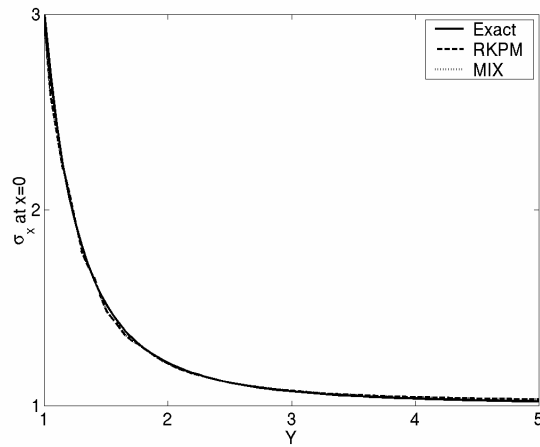


Figure 19 Stress comparison between RKPM and Mix RK Interpolation

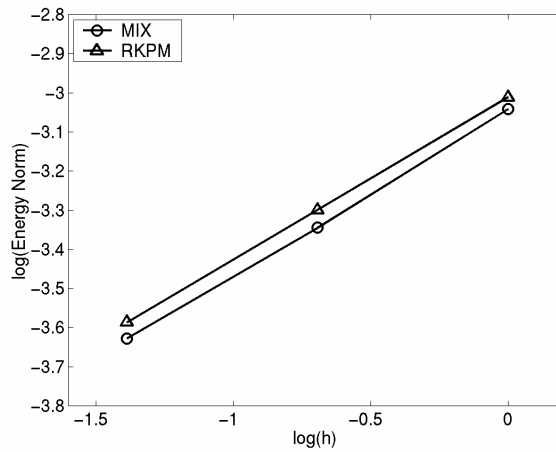


Figure 20 Energy norm comparison between RKPM and Mix RKI

7. Conclusions

A general formulation for constructing meshfree interpolation functions that preserve reproducing conditions is presented. It is based on a coupling of a primitive function and an enrichment function. The primitive function introduces Kronecker delta properties, while the enrichment function is responsible for meeting reproducing conditions. By enforcing reproducing conditions on the combined shape function, an orthogonality condition at nodal points exists between the vector of the enrichment functions and the vector of shifted monomial functions. It is shown that if the enrichment basis functions and the monomial basis functions form a regular pair, the orthogonality condition leads to a nullity of the enrichment function at nodal locations. As a consequence, the reproducing kernel shape function becomes an interpolation function.

Based on this result, a reproducing kernel interpolation function is constructed by coupling a nodal Kronecker delta primitive function and an enrichment function that takes the form of reproducing kernel approximation. The nodal Kronecker delta function can be easily constructed by normalizing any kernel functions used in the reproducing kernel approximation with the restricted support size so that it does not cover any neighbor points. This primitive function can be flexibly added to selected points or to all points for desired interpolation properties. For points with added primitive functions, the coefficients of the corresponding reproducing kernel shape functions become nodal values. This provides a convenient means for imposing essential boundary conditions. Optimal order error estimates are shown rigorously for the proposed reproducing kernel interpolant in any Sobolev norms. Optimal order convergence is maintained when the method employed to solve one-dimensional boundary value problems. In two-

dimensional problems, the order of convergence is not optimum. This is due to the fact that the essential boundary conditions are imposed only at discrete points. Similar phenomenon was observed in the meshfree solution using standard reproducing kernel approximation.

When the primitive functions are selected to be the finite element shape functions, the proposed method reduces to the approach by Huerta et al. [16], in which a background grid was needed. Since finite element shape functions are linearly complete, the basis functions in the enrichment function have to be at least second order. The smoothness of the resulting shape function is governed by the smoothness of the finite element shape function which is generally C^0 . In the proposed method, since the proposed primitive function is merely a normalized kernel function, basis functions with arbitrary order can be used in the enrichment function. The continuity of the resulting shape function is the minimum continuity of the primitive and enrichment functions. Therefore taking the same kernel function for primitive and enrichment functions leads to a shape function that has the same smoothness as the two component functions. Of course specific continuity in the primitive function can be chosen for particular applications.

The major advantage of the proposed method resides on its simplicity in dealing with essential boundary conditions or point loads without finite element enrichment. When a full reproducing kernel interpolation is employed, the rate of convergence is slightly lower than that of the regular reproducing kernel approximation. On the other hand, a mixed interpolation by introducing primitive functions only to the essential boundary points allows the standard treatment of essential boundary conditions while maintaining the good convergence properties of reproducing kernel approximation.

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