Grid Convergence in Several Simple Problems

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Grid Convergence in the AIAA Guidelines

- **"Most important activity in verification testing is systematically refining … grid size"**
- **"Objective … is to estimate discretization error"**

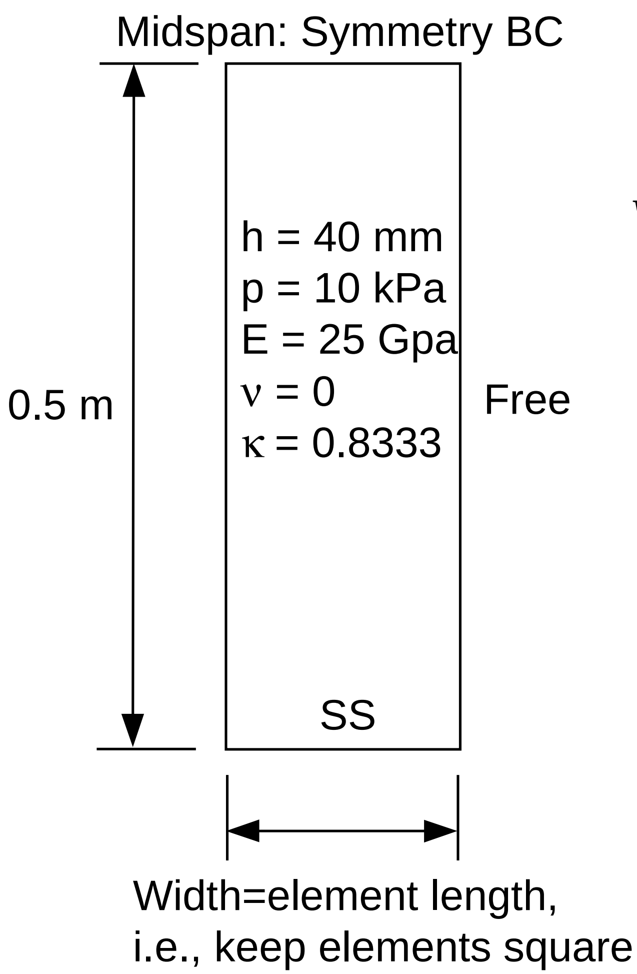
Digression

- If $w_{calc} = w_{exact} + a_1 h^p + O(h^q)$ as $h \to 0$, $q > p$ (1)

  then $w_{exact} = \dfrac{w_{calc2} h_1^p - w_{calc1} h_2^p}{h_1^p - h_2^p} + O\left(h^q\right)$ as $h \to 0$ (2)
- Richardson's extrapolation is the 1st term on the RHS of (2)

- **"Richardson's extrapolation can be used to estimate zero grid spacing [result]"**
  - **Requires "monotonic convergence"**
  - **Requires knowing order of convergence**
- **"Until computed convergence rate from 2 solutions … matches the known order of accuracy …, Richardson's extrapolation cannot be used to estimate error"**

Beam with Shear Deformation



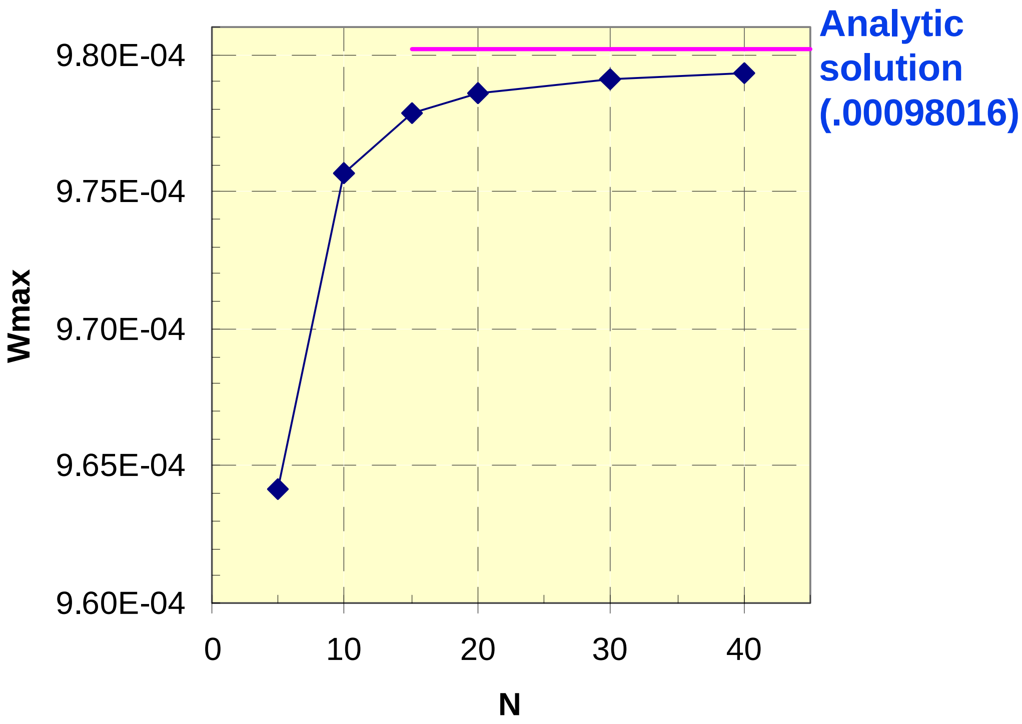
- **Exact solution:**

$$w_{mid} = \frac{5}{384}\left[1+\frac{8h^2}{5(1-\nu)\kappa^2 a^2}\right]\frac{12(1-\nu^2)pa^4}{Eh^3}$$
$$= 0.9801625\,\text{mm}$$

- **Model with shell elements**
- **In numerical solution, must constrain against in-plane rigid body motion. We imposed a symmetry condition across the left edge.**

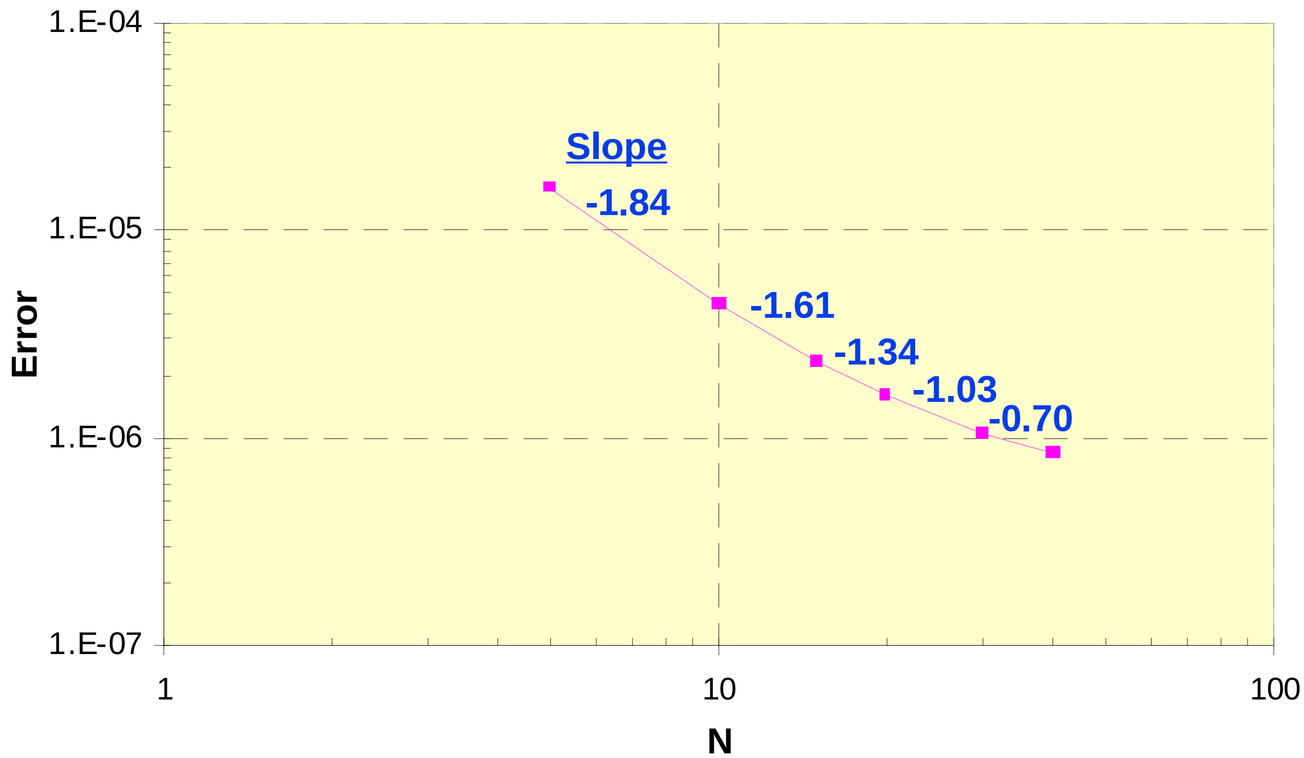
Example: Beam with Shear Deformation

- Analytic solution developed
- DYNA3D solutions obtained
  - Grids from 1X5 to 1X40
  - YASE shell elements
  - Elements held square, pressure held constant
  - Dynamic relaxation used to get static solution
- Max transverse deflection, at midspan, used as metric



Does it converge? Probably.
To the analytic solution? Maybe.

Beam Problem: Error Relative to Analytic Solution



Slope **should approach a constant but instead,**
error **seems to be it approaching a constant.**
What's going on?
May be converging, but not to expected solution.

Assessing Richardson Extrapolations

- Try Richardson extrapolation on successive solution pairs
  - Must know or assume order of convergence
  - If successive pairs give consistent zero-grid extrapolations, done
  - If not, must either refine grid further, or
- Use Eq. (3) with q=p+1, solve for zero-grid value, a, and b using successive triples of grid-refined solutions

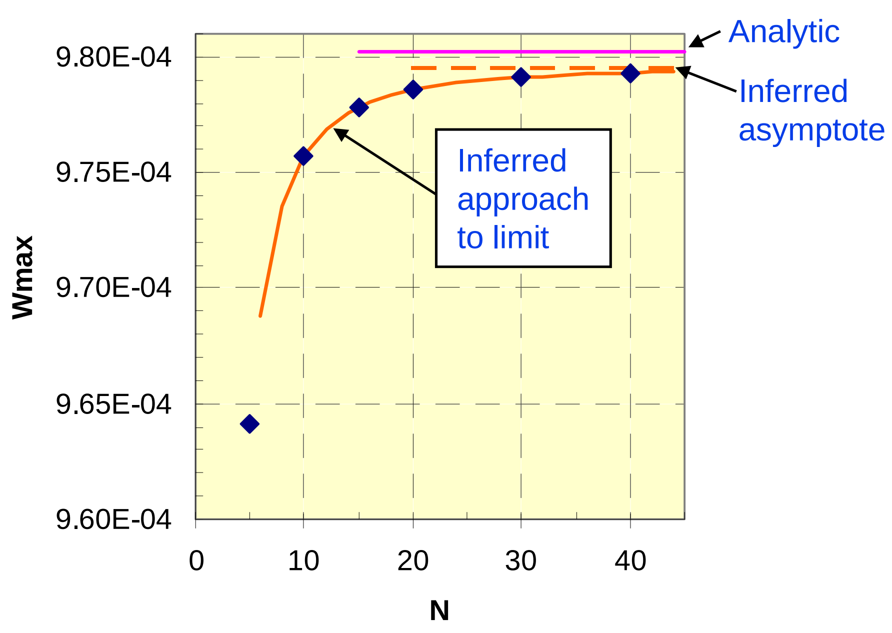
$$w_{calc} = w_{exact} + ah^p + bh^q + o(h^q) \quad \textbf{(3)}$$

- If successive triples give consistent zero-grid result, done
- If corresponding zero-grid result differs from a known solution, that indicates imperfect understanding of the problem the code is solving.

Beam Problem: Richardson Extrapolations

- Use the form $w_{calc_n} = w_{asymp} + aN_n^{-2}$
- Perform pairwise extrapolations
- Extrapolated values of w and a are roughly constant for all pairs
- Extrapolated w clearly differs from analytic value 9.8016E-4
- To reconcile, would need intimate knowledge of shell model and DYNA3D algorithm for it
- In this example, Richardson extrapolation works
  - Successive pairs give consistent results
  - Demonstrates imperfect knowledge of what problem code is solving
  - 3-pt extrapolation not necessary

N	*wasymp*1E4*	*a*1.E4*
5,10	9.79541	3.84767
10,15	9.79547	3.85380
15,20	9.79532	3.82114
20,30	9.79551	3.89520
30,40	9.79552	3.90857



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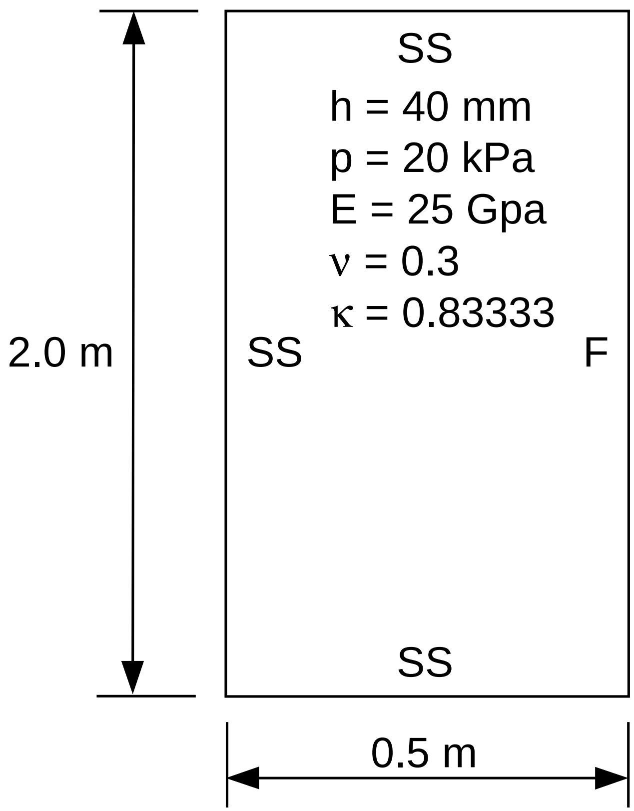
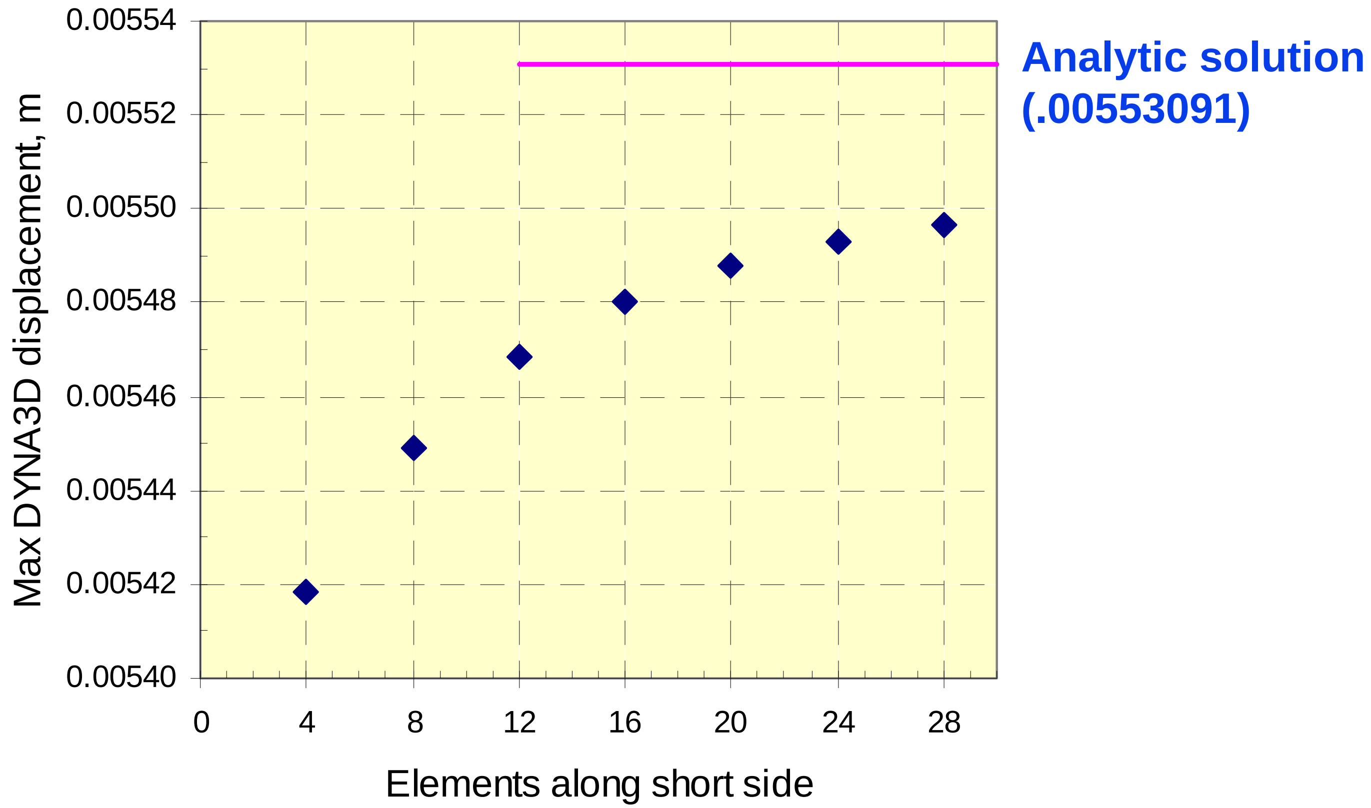
More Complex Example: SSSF Mindlin Plate



- **Exact solution developed**
  - **Mathematica used to get form suitable for numerical evaluation to arbitrary accuracy**
- **DYNA3D solutions obtained**
  - **Grids from 4X8 to 28X56**
  - **Dynamic relaxation used to get static solution**
  - **Same dynamic relaxation convergence parameters in all cases**
- **Max transverse deflection, at middle of free edge, used as metric**

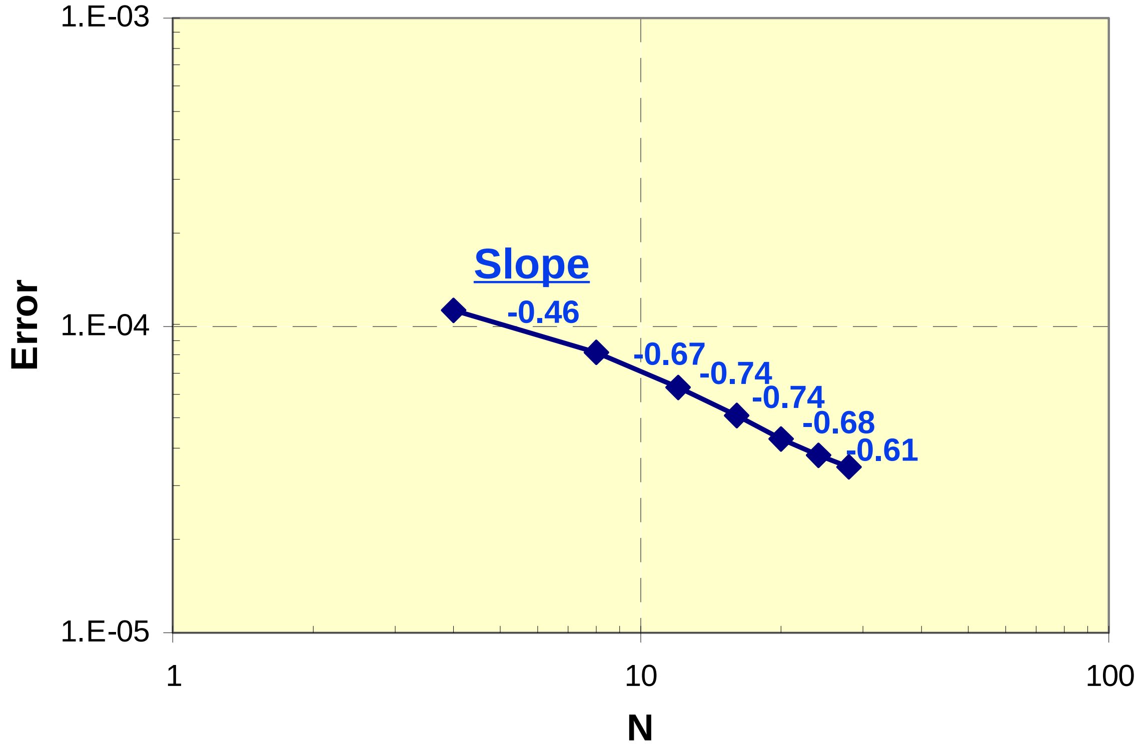
Plate Example: Raw Results



Does it converge? Probably.

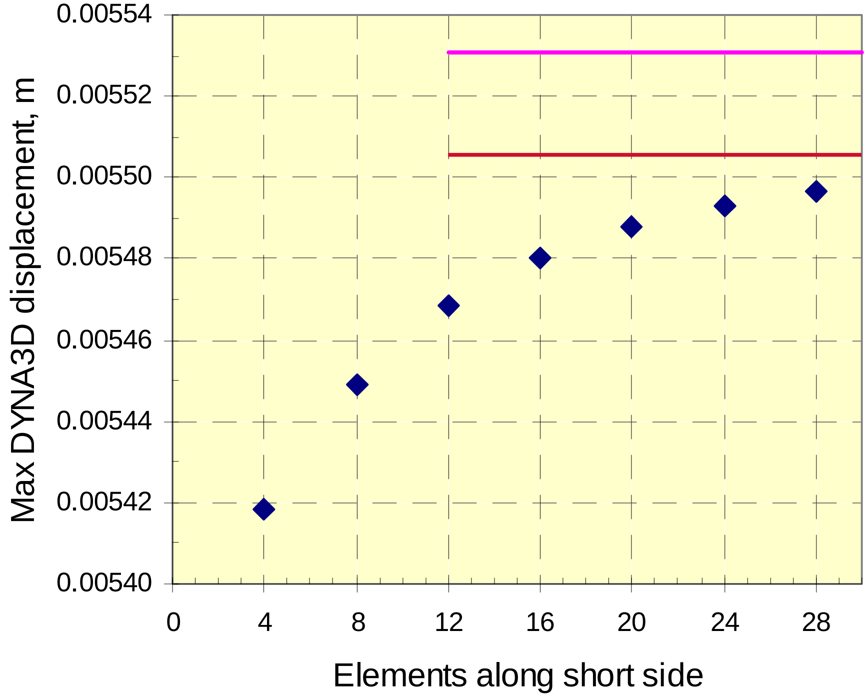
To the analytic solution? Probably not

Plate Example: Errors Relative to Analytic Solution



- **Inconclusive whether slope is approaching a constant**
- **If yes, then it's well less than 2, but these are the same type shell elements as in simple example**
- **Proceed with 2- and 3-point extrapolations**

Plate Example: Richardson Extrapolations



N	*wasymp*1.E3*	*a*1.E3*
4,8	5.4591	-0.6547
8,12	5.4841	-2.2499
12,16	5.4957	-3.9300
16,20	5.5017	-5.4613
20,24	5.5044	-6.5193
24,28	5.5058	-7.3382

- Unlike beam example, pairwise extrapolations inconsistent
- Same element formulation as in beam problem, so convergence order probably the same, i.e. 2
- Higher order terms must still be significant

Plate Example: 3-Point Extrapolations

- Assume $w_{calc_n} = w_{asymp} + aN_n^{-2} + bN_n^{-3}$
- Solve for w_{asymp} , a , b using 3 calculated results at a time

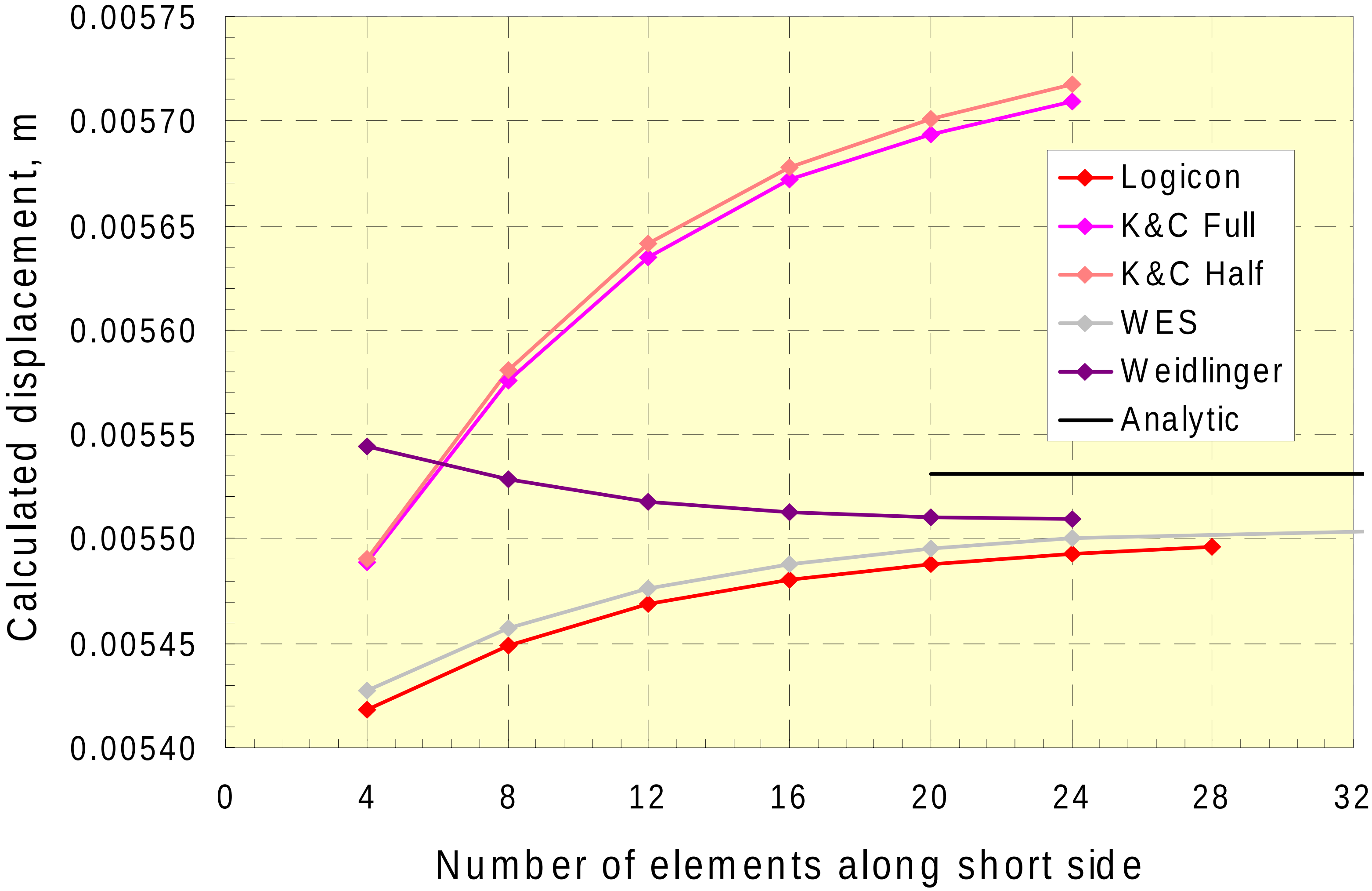
N	wasymp	a*1000	b*100	b/a/N
4,8,12	5.49030	-4.14408	1.19635	-0.241
8,12,16	5.50221	-7.76725	3.48467	-0.280
12,16,20	5.50695	-10.56988	6.02976	-0.285
16,20,24	5.50753	-11.14035	6.70310	-0.251
20,24,28	5.50795	-11.75220	7.59061	-0.231

- Consistency of these extrapolations suggests that Richardson failed due to significant 3rd order terms
- Last column shows 3rd order terms still not negligible compared to 2nd order, but may be declining in significance
- Disagreement with analytic solution must mean the numerical model has some features not in the analytic solution

Plate Problem Round Robin: Codes and Solution Methods

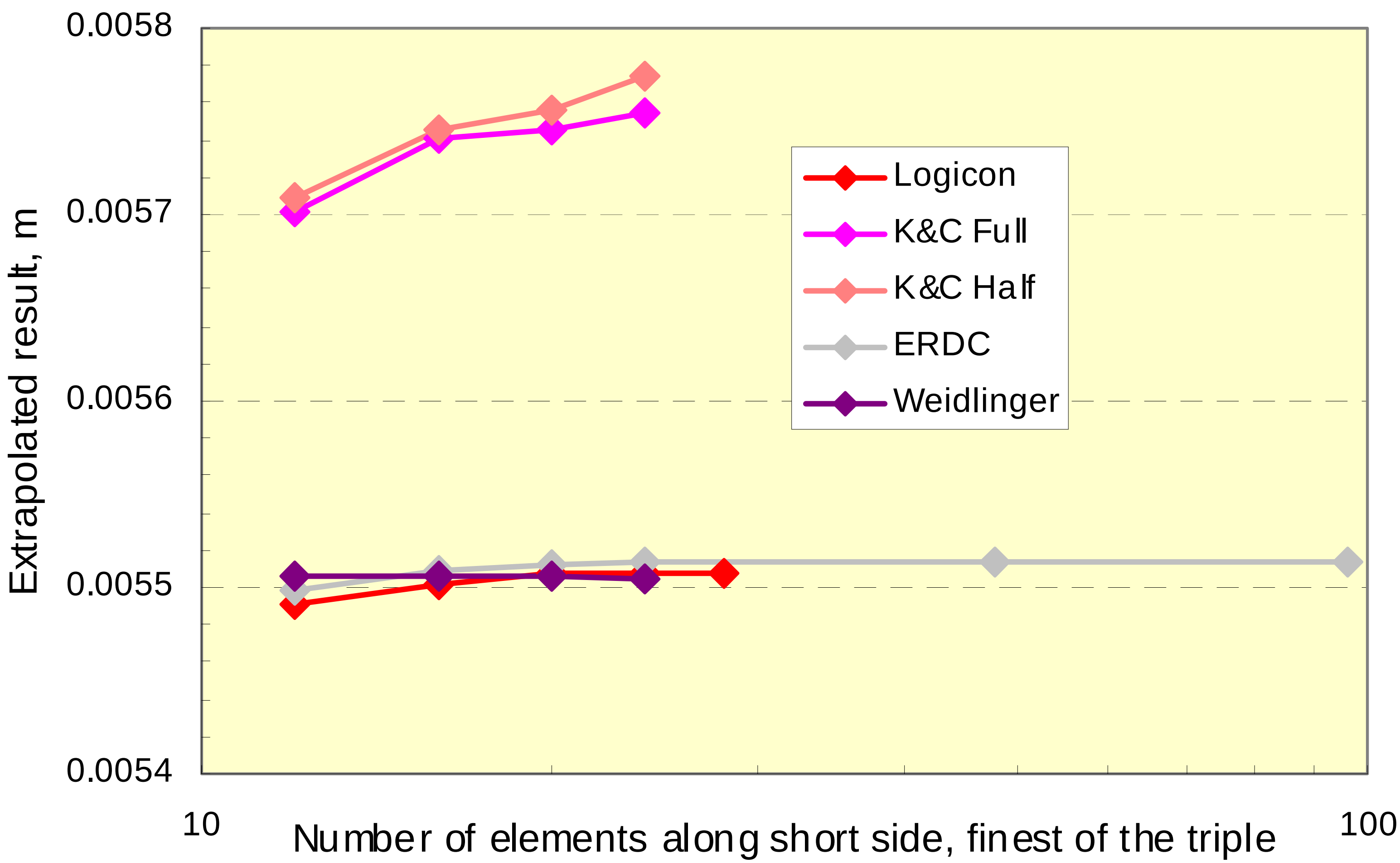
Data Set	Code	Element	Loading method	Constraint on z-rotation	Notes
ERDC	DYNA3D	YASE	0.25-ms cubic ramp, use mean over next 0.25 ms	Full	• Bathe-Dworkin element did not run. • Belytschko-Tsay did not run with finer grids. • Displacement varied less than 0.005 percent during 2nd 0.25 ms.
K&C Full	DYNA3D	YASE	Dynamic relaxation	None	• Used full model (no symmetry) • DR tolerance 1.0E-5 • Belytschko-Tsay and Hughes-Liu would not adequately converge
K&C Half	DYNA3D	YASE	Dynamic relaxation	None	• DR tolerance 1.0E-5
Logicon	DYNA3D	YASE	Dynamic relaxation	SS edges	• DR tolerance 1.0E-5 • Belytschko-Tsay would not converge to better than 1.0E-2 tolerance
Weidlinger	FLEX	Layered shell	Dynamic relaxation	None	• Converged to 5th significant figure • Small displacement formulation

Raw Results

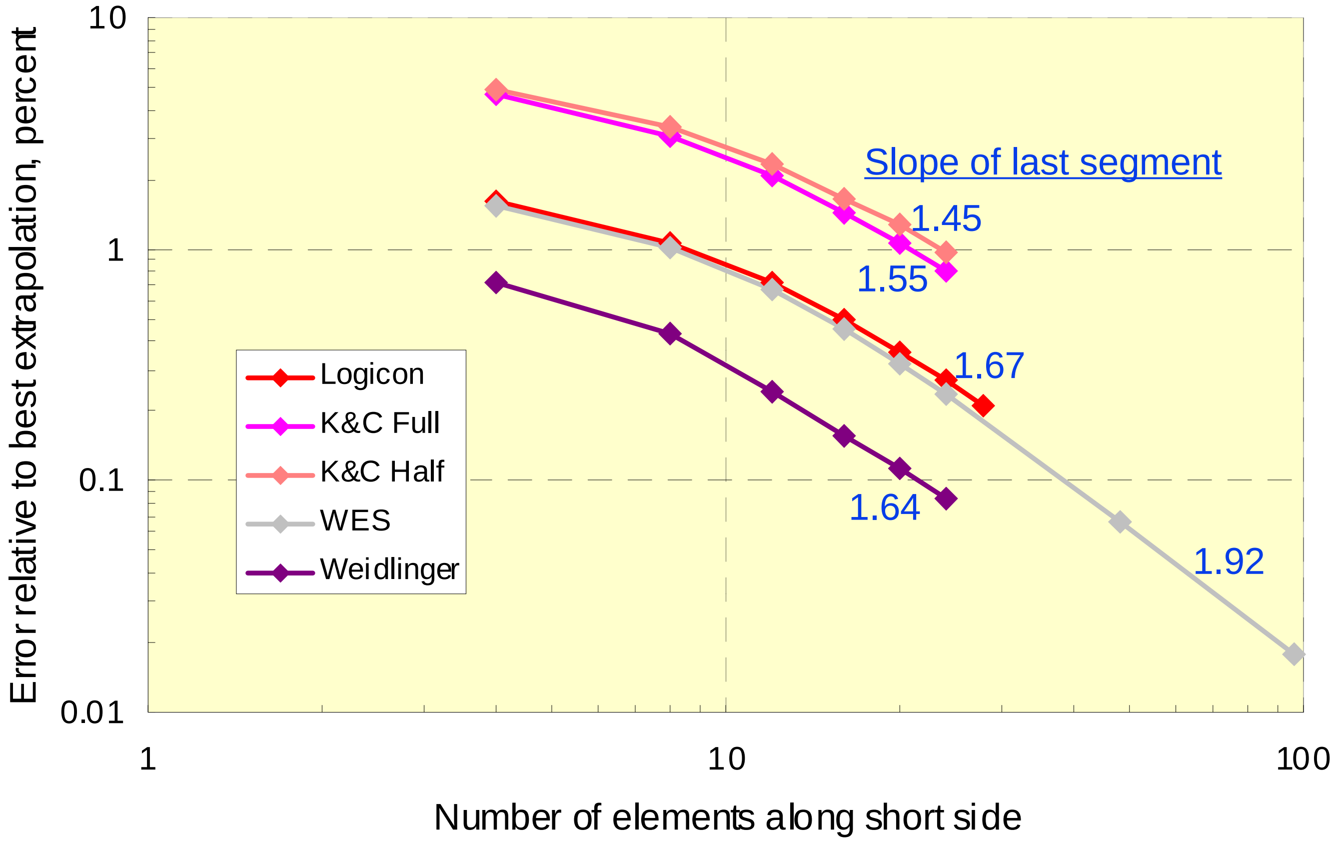


0.00575
0.00570
0.00565
0.00560
0.00555
0.00550
0.00545
0.00540
Calculated displacement, m
0
4
8
12
16
20
24
28
32
Number of elements along short side
Logicon
K&C Full
K&C Half
WES
Weidlinger
Analytic

Convergence of 3-point Extrapolations



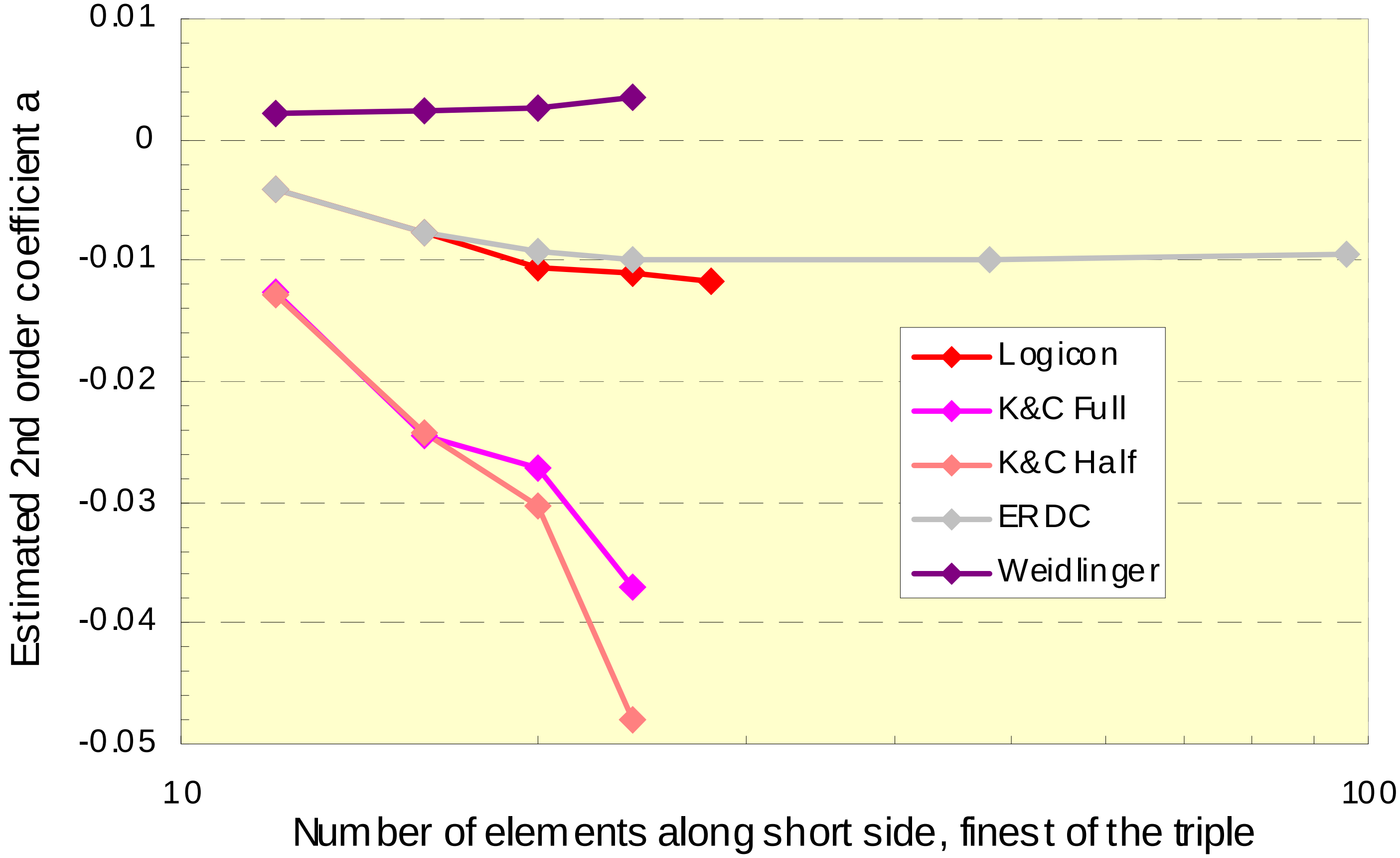
Errors Relative to Best Extrapolation



Slopes would all be -2 if Richardson extrapolation were valid, i.e., if 2nd-order errors dominated higher-order errors

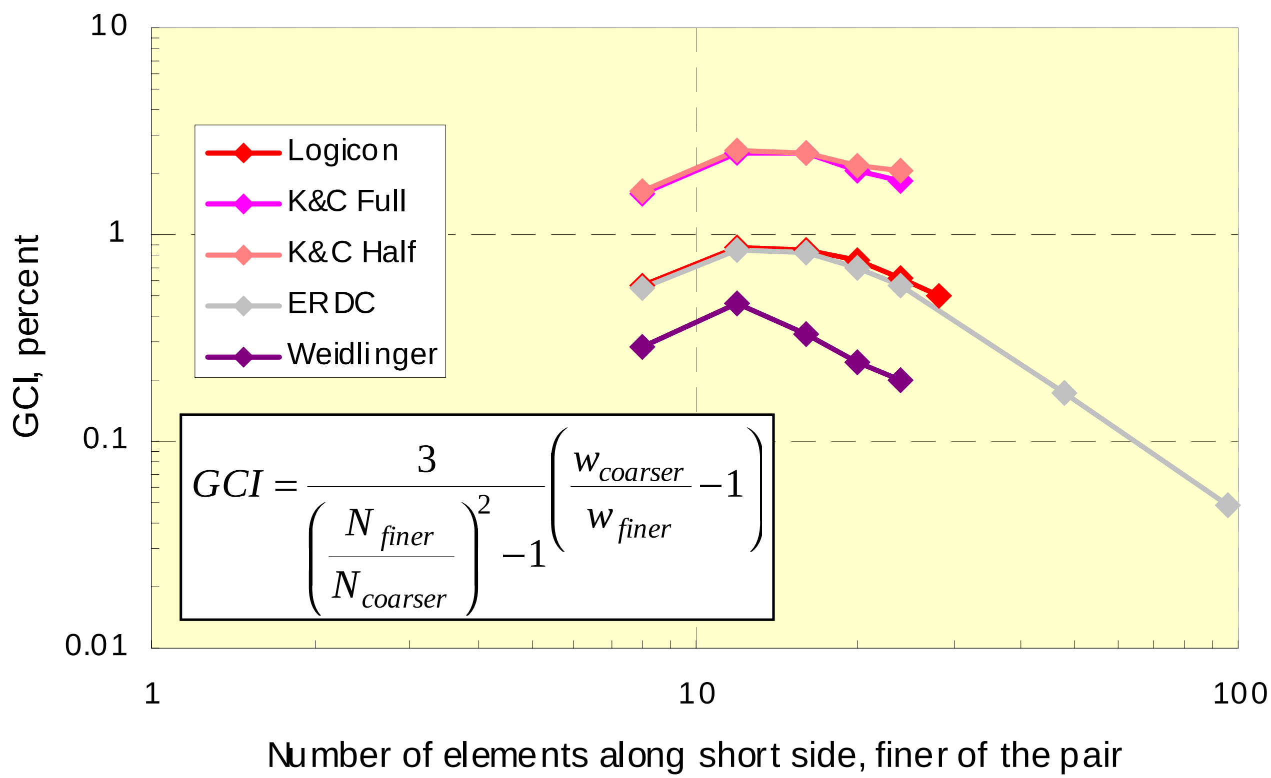
Estimates of 2nd Order Coefficient *a*

$$w_{calc_n} = w_{asymp} + aN_n^{-2} + bN_n^{-3}$$



Roache's Grid Convergence Index

- Proportional to fractional difference between finer-gridded calculation and Richardson's extrapolation
- Proportionality constant of 3 chosen so that for grid-doubling with 2nd-order method, GCI is exactly the fractional difference between finer and coarser-gridded solutions.



Conclusions

- In the two problems and six solutions considered, zero-grid extrapolations always disgreed with the expected analytic solutions
  - This implies that comparison of calculated and expected analytic solution in these cases give no information about order of convergence
- In the beam problem considered, Richardson (2-point) extrapolation was entirely adequate
  - Result from coarsest pair of grids (N=5,10) agreed with that from finest (30,40) within one digit in 5th significant figure
  - This implies that higher-order errors were always negligible compared with 2nd-order errors
- In the plate problem considered
  - Richardson extrapolation was not adequate, even up to N=96 (18432 elements in half of the plate)
  - 3-point extrapolations were adequate in 3 of 5 solutions, implying that convergence is 2nd order but 3rd-order terms were not negligible