Exact Solution for a Uniformly Loaded SSSF Mindlin Plate, Manipulated for Accurate Numerical Evaluation

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We present here a concise summary of an exact solution for transverse deformations of a uniformly loaded Mindlin plate with three edges simply supported and the fourth edge free. The solution is then rearranged into a form suitable for accurate numerical evaluation.

The relevant physical parameters are W (transverse displacement), Ψ_x, Ψ_y (rotations of original normals to middle surface), X (in-plane coordinate parallel to the free edge), Y (other in-plane coordinate), q (transverse load), b (length of the two parallel SS edges), a (other in-plane plate dimension), ν (Poisson's ratio), E (Young's modulus), h (plate thickness), k_s (shear deformation constant).

Dimensionless variables are defined as follows:

$$x = X/a \quad , \quad y = Y/a \quad , \quad \beta = b/a \quad ,$$

$$w = WD/qa^4 \quad , \quad \psi_x = (D/qa^3)\Psi_x \quad , \quad \psi_y = (D/qa^3)\Psi_y$$

$$D = Eh^3/[12(1-\nu^2)] \quad , \quad \varepsilon^2 = h^2/[6a^2k_s^2(1-\nu)]$$

In dimensionless variables, the PDE's to be solved are

$$\begin{aligned}
&\varepsilon^2\left[\frac{1-\nu}{2}\nabla^2\psi_x + \frac{1+\nu}{2}(\psi_{x,xx}+\psi_{y,xy})\right] - \psi_x - w_{,x} = 0 \quad ,\\
&\varepsilon^2\left[\frac{1-\nu}{2}\nabla^2\psi_y + \frac{1+\nu}{2}(\psi_{y,yy}+\psi_{x,xy})\right] - \psi_y - w_{,y} = 0 \quad ,\\
&\nabla^2 w + \psi_{x,x} + \psi_{y,y} = -\varepsilon^2 \quad , \quad -1/2 < x < 1/2 \quad , \quad 0 < y < \beta \quad ,
\end{aligned} \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad ,$$

and subscript commas denote partial differentiation. A general solution can be written in the form

$$
\begin{aligned}
w(x,y) &= \frac{5}{384} + \frac{\varepsilon^2}{8} - \left(\frac{1}{16} + \frac{\varepsilon^2}{2}\right)x^2 + \frac{1}{24}x^4 + \sum_{n=1}^{\infty} w_n(y)\cos\alpha_n x \quad , \\
\psi_x(x,y) &= \frac{x}{8} - \frac{x^3}{6} + \sum_{n=1}^{\infty} \psi_{xn}(y)\sin\alpha_n x \quad , \\
\psi_y(x,y) &= \sum_{n=1}^{\infty} \psi_{yn}(y)\cos\alpha_n x \quad ,
\end{aligned}
\tag{2}
$$

where $\alpha_n = (2n-1)\pi$. In the solution (2), the polynomial terms by themselves satisfy (1), so the other terms–the trigonometric sums–must satisfy the homogeneous form of (1), i.e., where the RHS of $(1)_3$ is set to zero.

The boundary conditions are as follows:

$$
\begin{aligned}
& w(\pm 1/2, y) = M_{xx}(\pm 1/2, y) = \psi_y(\pm 1/2, y) = 0 \quad , \quad 0 < y < \beta \quad , \\
& w(x,0) = M_{yy}(x,0) = \psi_x(x,0) = 0 \quad , \quad -1/2 < x < 1/2 \quad , \\
& M_{yy}(x,\beta) = M_{xy}(x,\beta) = Q_y(x,\beta) = 0 \quad , \quad -1/2 < x < 1/2 \quad ,
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
& M_{xx} = \psi_{x,x} + \nu\psi_{y,y} \quad , \quad M_{yy} = \psi_{y,y} + \nu\psi_{x,x} \quad , \\
& M_{xy} = \frac{1-\nu}{2}(\psi_{x,y} + \psi_{y,x}) \quad , \quad Q_y = \varepsilon^2(\psi_y + w_{,y}) \quad .
\end{aligned}
$$

The solution (2) exactly satisfies the *x*-boundary conditions $(3)_1$. The remaining work involves finding the forms of the *y*-dependent factors in the sums in (2) such that the PDE's (1) and *y*-boundary conditions $(3)_2$ and $(3)_3$ are satisfied.

Without going into detail, the *y*-dependent factors in the sums can be shown to take the forms

$$
\begin{aligned}
w_n(y) &= a_n \cosh\alpha_n y + b_n y\cosh\alpha_n y + c_n \sinh\alpha_n y + d_n y\sinh\alpha_n y \quad , \\
\psi_{yn}(y) &= -a_n\alpha_n \sinh\alpha_n y - b_n[(1+2\varepsilon^2\alpha_n^2)\cosh\alpha_n y + \alpha_n y\sinh\alpha_n y] \\
&\quad - c_n\alpha_n \cosh\alpha_n y - d_n[(1+2\varepsilon^2\alpha_n^2)\sinh\alpha_n y + \alpha_n y\cosh\alpha_n y] \\
&\quad - e_n\alpha_n \sinh\alpha'_n y + f_n\alpha_n \cosh\alpha'_n y \quad , \\
\psi_{xn}(y) &= a_n\alpha_n \cosh\alpha_n y + b_n(2\varepsilon^2\alpha_n^2 \sinh\alpha_n y + \alpha_n y\cosh\alpha_n y) \\
&\quad + c_n\alpha_n \sinh\alpha_n y + d_n(2\varepsilon^2\alpha_n^2 \cosh\alpha_n y + \alpha_n y\sinh\alpha_n y) \\
&\quad + e_n\alpha'_n \cosh\alpha'_n y - f_n\alpha'_n \sinh\alpha'_n y \quad ,
\end{aligned}
\tag{4}
$$

where

$$
\alpha'_n = \sqrt{\alpha_n^2 + \frac{2}{(1-\nu)\varepsilon^2}}
$$

and where $(a_n\ , b_n\ , c_n\ , d_n\ , e_n\ ,\ f_n)$ are constants to be determined by the *y*-boundary conditions.

To deal with the *y*-boundary conditions we must first expand each *x*-polynomial in (2) in a Fourier series on the interval (-1/2,1/2) so they can be combined termwise with the trigonometric sums. To this end we write

$$x^p = \begin{cases} \sum_{n=1}^{\infty} r_{pn} \cos\alpha_n x \quad , \quad p \text{ even} \\ \sum_{n=1}^{\infty} r_{pn} \sin\alpha_n x \quad , \quad p \text{ odd} \end{cases} \quad , \quad -1/2 < x < 1/2$$

By standard means we find

$$r_{0n} = \frac{4(-1)^{n+1}}{\alpha_n} \quad , \quad r_{1n} = \frac{4(-1)^{n+1}}{\alpha_n^2} \quad , \quad r_{2n} = \frac{(-1)^{n+1}}{\alpha_n}\left(1 - \frac{8}{\alpha_n^2}\right) \quad ,$$
$$r_{3n} = \frac{3(-1)^{n+1}}{\alpha_n^2}\left(1 - \frac{8}{\alpha_n^2}\right) \quad , \quad r_{4n} = \frac{(-1)^{n+1}}{\alpha_n}\left(\frac{1}{4} - \frac{12}{\alpha_n^2} + \frac{96}{\alpha_n^4}\right) \quad . \tag{5}$$

So from Eqs. (2) and (5) we can write

$$w(x,y) = \sum_{n=1}^{\infty} [g_n + w_n(y)] \cos\alpha_n x \quad ,$$
$$\psi_x(x,y) = \sum_{n=1}^{\infty} [h_n + \psi_{xn}(y)] \sin\alpha_n x \quad , \tag{6}$$
$$\psi_{x,x}(x,y) = \sum_{n=1}^{\infty} [j_n + \alpha_n \psi_{xn}(y)] \cos\alpha_n x \quad ,$$

provided

$$g_n = \left(\frac{5}{384} + \frac{\varepsilon^2}{8}\right) r_{0n} - \left(\frac{1}{16} + \frac{\varepsilon^2}{2}\right) r_{2n} + \frac{1}{24} r_{4n} \quad ,$$
$$h_n = \frac{1}{8} r_{1n} - \frac{1}{6} r_{3n} \quad ,$$
$$j_n = \frac{1}{8} r_{0n} - \frac{1}{2} r_{2n} \quad .$$

Now we are in position to deal with the BC's $(3)_2$ at $y = 0$. Using (2), (4) and (6), and setting each term of the trigonometric sum to zero, we find the following three equations for each *n*:

$$\begin{aligned}
&g_n + a_n = 0 \quad , \\
&\nu j_n - (1-\nu)\alpha_n^2 a_n - 2\alpha_n[1+(1-\nu)\varepsilon^2\alpha_n^2]d_n - (1-\nu)\alpha_n\alpha_n' e_n = 0 \quad , \\
&h_n + \alpha_n a_n + 2\varepsilon^2\alpha_n^2 d_n + \alpha_n' e_n = 0 \quad .
\end{aligned} \tag{7}$$

The unknowns here are a_n , d_n , and e_n . The equations can be solved either numerically or symbolically.

Equations for the other three sets of constants come from the BC's $(3)_3$ at $y = \beta$. By again using (2), (4), and (6), we find the following three equations for each n:

$$\begin{aligned}
&\nu j_n - a_n(1-\nu)\alpha_n^2 \cosh\alpha_n\beta - c_n(1-\nu)\alpha_n^2 \sinh\alpha_n\beta \\
&\quad + b_n\{-2[1+(1-\nu)\varepsilon^2\alpha_n^2]\alpha_n \sinh\alpha_n\beta - (1-\nu)\alpha_n^2\beta\cosh\alpha_n\beta\} \\
&\quad + d_n\{-2[1+(1-\nu)\varepsilon^2\alpha_n^2]\alpha_n \cosh\alpha_n\beta - (1-\nu)\alpha_n^2\beta\sinh\alpha_n\beta\} \\
&\quad - e_n\alpha_n\alpha_n'(1-\nu)\cosh\alpha_n'\beta + f_n\alpha_n\alpha_n'(1-\nu)\sinh\alpha_n'\beta = 0 \quad , \\
&a_n\alpha_n^2 \sinh\alpha_n\beta + c_n\alpha_n^2\cosh\alpha_n\beta \\
&\quad + b_n[\alpha_n(2\varepsilon^2\alpha_n^2+1)\cosh\alpha_n\beta + \alpha_n^2\beta\sinh\alpha_n\beta] \\
&\quad + d_n[\alpha_n(2\varepsilon^2\alpha_n^2+1)\sinh\alpha_n\beta + \alpha_n^2\beta\cosh\alpha_n\beta] \\
&\quad + e_n[\alpha_n^2 + 1/\varepsilon^2(1-\nu)]\sinh\alpha_n'\beta - f_n[\alpha_n^2 + 1/\varepsilon^2(1-\nu)]\cosh\alpha_n'\beta = 0 \quad , \\
&-2b_n\varepsilon^2\alpha_n^2\cosh\alpha_n\beta - 2d_n\varepsilon^2\alpha_n^2\sinh\alpha_n\beta \\
&\quad - e_n\alpha_n\sinh\alpha_n'\beta + f_n\alpha_n\cosh\alpha_n'\beta = 0 \quad .
\end{aligned} \tag{8}$$

After having solved Eqs. (7), the unknowns in (8) are b_n , c_n , and f_n . Once Eqs. (8) are solved the problem is solved, as expression (2) and (4) can be evaluated completely.

By using the symbolic manipulation facilities of Mathematica we have found the following expressions for the constants in the expression for the transverse displacement:

$$\begin{aligned}
&a = 4(-1)^n(1+\varepsilon^2\alpha^2)\alpha^{-5} \quad , \quad d = -2(-1)^n\alpha^{-4} \quad , \\
&b = -4(-1)^n N_b\alpha^{-4}\Delta^{-1} \quad , \quad c = -4(-1)^n N_c(1-\nu)^{-1}\alpha^{-5}\Delta^{-1} \quad ,
\end{aligned} \tag{9}$$

where the subscript n on a, b, c, d, and α has been omitted, and where

$$\begin{aligned}
&N_b = [3-\nu+(3+\nu)C](C-1)C' - 2(1-\nu)\alpha\alpha'\varepsilon^2 SCS' + 2(1-\nu)\alpha^2\varepsilon^2 S^2 C' \quad , \\
&N_c = -2[3-\nu+(3-2\nu-\nu^2)C](C-1)C' + 2(1-\nu)\alpha[\beta\nu SC' + 2(1-\nu)\varepsilon^2\alpha' SCS'] \\
&\qquad + (1-\nu)\alpha^2[(1-\nu)\beta^2 C' - 2(5-\nu)\varepsilon^2 S^2 C' - 2(1-\nu)\varepsilon^2\beta\alpha' S'] \\
&\qquad + 4(1-\nu)^2\alpha^3\alpha'\varepsilon^4 SCS' - 4(1-\nu)^2\alpha^4\varepsilon^4 S^2 C' \quad , \\
&\Delta = -2(3+\nu)SCC' - 2(1-\nu)\alpha[\beta C' - 2\varepsilon^2\alpha' C^2 S'] - 4(1-\nu)\alpha^2\varepsilon^2 SCC' \quad ,
\end{aligned} \tag{10}$$

with the shorthand

$$S = \sinh\alpha\beta \quad , \quad C = \cosh\alpha\beta \qquad S' = \sinh\alpha'\beta \quad , \quad C' = \cosh\alpha'\beta \quad .$$

The arguments of the hyperbolic functions will grow linearly with n. This will cause problems when we numerically evaluate the displacements, so we will rewrite the solution to avoid large arguments. When we expand the hyperbolics as exponentials, we see that each of the three quantities defined in (10) is dominated by the factor $e^{(2\alpha+\alpha')\beta}/8$, so we write

$$N_b = \frac{e^{(2\alpha+\alpha')\beta}}{8} N_b^* \quad , \quad N_c = \frac{e^{(2\alpha+\alpha')\beta}}{8} N_c^* \quad , \quad \Delta = \frac{e^{(2\alpha+\alpha')\beta}}{8} \Delta^* \quad ,$$

where

$$\begin{aligned}
N_b^* &= [2(3-\nu)\sqrt{E} + (3+\nu)(1+E)](1+E-2\sqrt{E})(1+E') \\
&\quad - 2(1-\nu)\varepsilon^2\alpha\alpha'(1-E^2)(1-E') + 2(1-\nu)\varepsilon^2\alpha^2(1-E)^2(1+E') \quad , \\
N_c^* &= -2[2(3-\nu)\sqrt{E} + (3-2\nu-\nu^2)(1+E)](1+E-2\sqrt{E})(1+E') \\
&\quad + 2(1-\nu)\alpha[2\beta\nu\sqrt{E}(1-E)(1+E') + 2(1-\nu)\varepsilon^2\alpha'(1-E^2)(1-E')] \\
&\quad + (1-\nu)\alpha^2[4(1-\nu)\beta^2 E(1+E') - 2(5-\nu)\varepsilon^2(1-E)^2(1+E') \qquad\qquad (11) \\
&\qquad\quad - 8(1-\nu)\varepsilon^2\beta\alpha' E(1-E')] \\
&\quad + 4(1-\nu)^2\varepsilon^4\alpha^3\alpha'(1-E^2)(1-E') - 4(1-\nu)^2\varepsilon^4\alpha^4(1-E)^2(1+E') \quad , \\
\Delta^* &= -2(3+\nu)(1-E^2)(1+E') - 2(1-\nu)\alpha[4\beta E(1+E') - 2\varepsilon^2\alpha'(1+E)^2(1-E')] \\
&\quad - 4(1-\nu)\varepsilon^2\alpha^2(1-E^2)(1+E') \quad ,
\end{aligned}$$

and where

$$E = e^{-2\alpha\beta} \quad , \quad E' = e^{-2\alpha'\beta} \quad .$$

Now from (4) we break the n^{th} term in the displacement sum into two parts:

$$w_n(y) = w_{acn}(y) + w_{bdn}(y) \quad ,$$

where

$$w_{acn}(y) = \frac{a+c}{2}e^{\alpha y} + \frac{a-c}{2}e^{-\alpha y} \quad , \quad w_{bdn}(y) = y\left(\frac{b+d}{2}e^{\alpha y} + \frac{b-d}{2}e^{-\alpha y}\right) \quad .$$

Now after extensive algebraic manipulations, Mathematica produces the following:

$$\begin{aligned}
\frac{b+d}{2} &= -\frac{2(-1)^n}{\alpha^4\Delta^*}\left(N_b^* + \frac{\Delta^*}{2}\right) \\
&= -\frac{2(-1)^n}{\alpha^4\Delta^*}\{-2\sqrt{E}(1+E')(1-\sqrt{E})[2\nu + (3-\nu)\sqrt{E} + (3+\nu)E] \\
&\qquad - 4\varepsilon^2\alpha^2(1-\nu)E(1-E)(1+E') \\
&\qquad - 4\alpha\beta(1-\nu)E(1+E') + 4\varepsilon^2\alpha\alpha'(1-\nu)E(1+E)(1-E')\} \quad ,
\end{aligned}$$

$$\begin{aligned}\frac{b-d}{2} = -\frac{2(-1)^n}{\alpha^4\Delta^*}\{&2(1+E')(1-\sqrt{E})[3+\nu+(3-\nu)\sqrt{E}+2\nu E] \\ &+4\varepsilon^2\alpha^2(1-\nu)(1-E)(1+E') \\ &+4\alpha\beta(1-\nu)E(1+E')-4\varepsilon^2\alpha\alpha'(1-\nu)(1+E)(1-E')\}\quad ,\end{aligned}$$

$$\begin{aligned}\frac{a+c}{2} = -\frac{2(-1)^n}{\alpha^5(1-\nu)\Delta^*}\Big[&-4\nu\sqrt{E}\{1+\nu-\alpha\beta(1-\nu)\}(1+E') \\ &+4E(1+E')\{3+\nu^2+\alpha^2(1-\nu)[(5-\nu)\varepsilon^2+(1-\nu)\beta^2]+2\alpha\beta(1-\nu)^2 \\ &\qquad +2\alpha^3\beta\varepsilon^2(1-\nu)^2+2\alpha^4\varepsilon^4(1-\nu)^2\} \\ &-8E\alpha\alpha'\varepsilon^2(1-E')(1-\nu)^2(1+\alpha\beta+\alpha^2\varepsilon^2) \\ &-4\nu E^{3/2}[1+\nu+\alpha\beta(1-\nu)](1+E')-4E^2(1+\alpha^2\varepsilon^2)(1-\nu)\{3+\nu \\ &\qquad +2\varepsilon^2\alpha(\alpha+\alpha')(1-\nu)+E'[3+\nu+2\varepsilon^2\alpha(\alpha-\alpha')(1-\nu)]\}\Big]\end{aligned}$$

$$\begin{aligned}\frac{a-c}{2} = -\frac{2(-1)^n}{\alpha^5(1-\nu)\Delta^*}\Big[&4(1+\alpha^2\varepsilon^2)(1-\nu)\{3+\nu+2\varepsilon^2\alpha(\alpha+\alpha')(1-\nu)+E'[3+\nu+2\varepsilon^2\alpha(\alpha-\alpha')(1-\nu)]\} \\ &+4\nu\sqrt{E}\{1+\nu-\alpha\beta(1-\nu)\}(1+E') \\ &-4E(1+E')\{3+\nu^2+\alpha^2(1-\nu)[(5-\nu)\varepsilon^2+(1-\nu)\beta^2]-2\alpha\beta(1-\nu)^2 \\ &\qquad -2\alpha^3\beta\varepsilon^2(1-\nu)^2+2\alpha^4\varepsilon^4(1-\nu)^2\} \\ &-8E\alpha\alpha'\varepsilon^2(1-E')(1-\nu)^2(1-\alpha\beta+\alpha^2\varepsilon^2) \\ &+4\nu E^{3/2}[1+\nu+\alpha\beta(1-\nu)](1+E')\Big]\end{aligned}$$