Constitutive Model V&V for Softening Materials

Don Simons
Logicon

21 May 2001
Revised 5 July 2001

Softening is a property of a constitutive model in which a decrease in stress accompanies an increase in strain. It is easy to identify in one-dimensional load paths simply by looking at the stress-strain curve. In multidimensional straining, the definition can be extended based on the loss of positive-definiteness in the matrix of incremental moduli.

Softening poses many difficulties for the structural analysts. The most significant is that there is no uniformly accepted way of treating it in Computational Structural Dynamic (CSD) codes based on continuum mechanics. But it cannot be ignored because from a practical viewpoint, evidence of it is commonly seen in laboratory tests. For example, in strain-controlled unconfined compression tests of plain concrete cylinders, engineering stresses soften as a function of engineering strain.

From the mathematical viewpoint, if softening is represented simply by a falling stress-strain curve in a rate-independent continuous material, then the partial differential equations of motion or equilibrium will change characteristic type at the onset of softening, from hyperbolic to elliptic in dynamic problems and the reverse in statics. In either case, the problem will become ill-posed, as the boundary and initial conditions for one class of equations are not appropriate for the other. As a result, it is impossible for rate-independent softening to proceed within any non-zero volume of the domain of solution. The only way to continue the solution is to introduce a “localization” of all softening onto a surface. Displacements will be discontinuous across this surface, and in the surrounding continuous regions, there may be “unloading”—decreasing stress with decreasing strain—but no softening.

The localization naturally has a useful physical interpretation as a failed region. But before making practical use of that interpretation, the analyst must consider whether his equations and his numerical implementation of them are being properly solved. If the constitutive model is rate-independent and local, then no numerical solution method designed for positive-definite materials will be valid when softening begins. Without special attention, a code may still run and produce an answer in which softening is confined to a thin region, perhaps only one element or zone wide. For such results to be useful and interpretable, one or more of the following methods---sometimes known as “localization limiters”---should be used:

1. Include rate-dependence in the constitutive model[1]. This may be either physical or numerical. It introduces higher temporal derivatives into the equations of motion, thereby resolving some of the issues of ill-posedness in the face of softening. Linear or quadratic artificial viscosity, or true material rate dependence such as e.g. Duvaut-Lions viscoplasticity, will suffice.
2. Make stresses depend on strain gradients[2]. Cosserat elasticity is the best-known example of this. It introduces higher spatial gradients into the equations of motion,

---

[1] Needleman, A. , “Material Rate Dependence and Mesh Sensitivity in Localization Problems,” *Comp Meth in Appl Mech and Engineering* **63** (1988) 69-85.

[2] Aifantis, E. C., “The Physics of Plastic Deformation,” *Int J Plasticity* **3** (1987) 211-247.

again resolving some of the ill-posedness issues. This is very difficult to program in the finite element method, since strain gradients are not normally computed or stored.

3. Include non-local material behavior, i.e., make the stresses at a point depend on the strains in a non-vanishing region around the point[3]. Mathematically this is similar to gradient-dependence, as one could regard the strains in the region around a point as a Taylor series whose coefficients include higher gradients of strain at the central point. This is also very difficult to program, because the constitutive model subroutines must query surrounding elements
4. Ensure proper internal energy dissipation during failure[4]. Fracture mechanics testing can quantify the "energy release rate," which is the energy required to produce a unit area of crack surface. During complete softening to zero stress, the constitutive model will dissipate energy per unit volume equal to the area beneath the stress-strain curve. To relate the two, one assumes that softening localizes into a region one element wide. Then the constitutive failure energy multiplied by the volume of the failed element must equal the specified energy release rate multiplied by the projected area of the putative crack through the element in question. This implies that the softening portion of the stress-strain curve must be adjusted according to the element size, becoming shallower as elements become smaller. Opinions are sharply divided on the legitimacy of this approach. On one hand it does preserve essential energetic relationships, but purists balk at making the constitutive model depend on element size.
5. Change mesh topology in anticipation of material failure[5]. This option clearly extends beyond the bounds of material modeling. But it is a possible direct approach to treating localization and failure. One could track conditions at the interfaces between elements rather than within the elements. If conditions for the onset of softening are met, then the two adjoining elements could be split apart along that boundary. At the same time, some mechanism must be introduced to dissipate the proper amount of energy in softening as stresses drop to zero. This might take the form of a simplified, temporary bridge element between the separating surfaces.

In light of the lack of consensus on treatment of softening in CSD codes, it is impossible to provide definitive guidelines here. One approach that is definitely unacceptable is the use of a rate-independent model with a decreasing stress-strain curve in the absence of any other special features to account for the softening. After selecting a localization limiter, one must still exercise great care in setting material parameters. Both physical evidence of softening and experimental determination of fracture energies come from laboratory structural tests with inhomogeneous deformations. This implies that some level of calibration—involving iterative calculations of one or more experimental setups—is virtually essential. Once calibrated, constitutive models with softening should be verified by following the same steps as outlined for all constitutive models. But in addition, the Verification suite should include grid convergence studies of problems with nominally homogeneous stress states. In this and in the Validation phase, special care must be taken not to reuse results that were used earlier for calibration.

---

[3] Bazant, Z., "Mechanics of Distributed Cracking," *Appl Mech Rev* **39** (1986) 675-705.

[4] Pijaudier-Cabot, G. et al, "Comparison of Various Models for Strain-Softening," *Eng Comput* **5** (1988) 141-150.

[5] Hillerborg, A., et al., "Analysis of Crack Formation and Crack Growth in Concrete by Means of Fracture Mechanics and Finite Elements," *Cement and Concrete Research* **6** (1976) 773-782.